

Seminar: Infinite Ramsey Theory

Lecturer

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Notes

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These are my notes on the seminar “Infinite Ramsey Theory”, taught by PROF. DR. ALEKSANDRA KWIATKOWSKA in the summer term 2024 at the University Münster.

Warning 0.1. This is not an official script. The official notes can be found [here](#).

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1 A Couple of Proofs of the van der Waerden Theorem

1st talk, A. KWIATKOWSKA, 2024-04-08

TODO: Copy from official notes

2 The Polynomial van der Waerden Theorem

2nd talk, R. SULLIVAN, 2024-04-15

Definition 2.1. We say that $p \in \mathbb{Z}[x]$ is **iNtEgRaL**^a iff $p(0) = 0$.

^ain the paper, they were called **integral**, but this is clearly bad name.

Find a better name

Theorem 2.2 (Polynomial van-der-Waerden). Let $p_1(x), \dots, p_m(x)$ be iNtEgRaL polynomials. Let $k \geq 1$. Then there exists $N \geq 1$ such that for any k -colouring of $[N]$, there exist $a, d \neq 0$, such that $a, a + p_1(d), \dots, a + p_m(d)$ have the same color.

The classical van-der-Waerden can be recovered by setting $p_j(x) := j \cdot x$. Classical van-der-Waerden is also called linear van-der-Waerden.

Remark 2.2.1. This is not true for $p(x)$ with $p(0) \neq 0$: Let $p(x) = 2x + 1$ and colour \mathbb{N}_+ by parity.

Originally, **Theorem 2.2** was proved using ergodic theory. We will use similar techniques of the proof of classical van-der-Waerden. Recall: Color-focussing, double induction.

Definition 2.3. Let $P = \{p_1(x), \dots, p_m(x)\}$ be a set of iNtEgRaL polynomials. We say that $\bar{b} \in \mathbb{Z}^m$ has **focus** $a \in \mathbb{Z}$ if $\exists d \neq 0. \forall 1 \leq j \leq m. b_j = a + p_j(d)$.

Remark 2.3.2. Note that this is the other way round to last week’s proof.

For sets of polynomials P , it is not clear what “next” means.

Definition 2.4. Let $P = \{p_1(x), \dots, p_m(x)\}$. Fix a k -coloring of $[N]$. Let $\bar{b}_1, \dots, \bar{b}_r \in [N]^m$. We say $\bar{b}_1, \dots, \bar{b}_r$ are **color-focussed** at $a \in \mathbb{Z}$ if

- a is a focus of each \bar{b}_i and
- each \bar{b}_i is monochromatic and the \bar{b}_i are of different colours.

Proof of Theorem 2.2 (Walters (98?)) We will first do the case of $p(x) = x^2$. For classical van-der-Waerden the strategy was to do

- an outer induction on the length of the arithmetic progression and
- an inner induction using colour-focussing and blocks.

Since we only consider one polynomial, it suffices to do (ii).

We show, by induction on r :

$$\forall 0 \leq r \leq k. \exists N. \forall k\text{-colourings of } [N] \tag{1}$$

$$\exists \text{ monochromatic } \{a, a + d^2\} \text{ or } r \text{ colour-focussed } \{a + d_i^2\}_{0 \leq i \leq r} \text{ together with focus } a$$

The cases $r = 0, 1$ are trivial.

Let $r = 2$. Let N witness Equation 2 for $n = 1$. We apply the same idea as in the linear van-der-Waerden. Use the induction hypothesis to get identically coloured blocks of size N and find blocks correctly located so that we can do “jumps”.

Missing picture, f focus

We want $f, \lambda, \mu, q(x)$ such that

$$f + p(\lambda) = a,$$

$$f + p(\mu) = a + p(d) + Nq(t).$$

So $q(t) = \frac{1}{N}(p(\mu) - p(\lambda))$. If this is integer and linear, we can use linear van-der-Waerden for this. So take $\mu = Nt + d, \lambda = Nt$,

$$q(t) = \frac{1}{N} (p(Nt + d) - p(Nt)) = 2dt.$$

Problem: d depends on the particular colouring of the block B and can't be fixed in advance. However, we know that d is bounded ($d \leq N$). So use linear van-der-Waerden with the last term of the arithmetic progression $\leq s + 2Nt$. Once we look in B_s and know d , we can fix $q(t) = 2dt$ and then find $B_{s+q(t)}$ coloured identically to B_s .

So we are now basically done: Given N take N' large enough using linear van-der-Waerden to guarantee that $B_s, B_{s+t}, \dots, B_{s+2N't}$ coloured identically where each block is of size N .

There is one small issue: maybe $f = 0$. But there is an easy fix:

Everything we did was translation invariant. $f = a - p(Nt)$, $t \leq N'$. So take $N' + p(N \cdot N')$.

For general r , for each $a + d_i^2$ use $q_{d_i}(t) = 2d_it$.

The only place where we used $p(x) = x^2$, was when applying linear van-der-Waerden.

In the general case we need to set up some induction. □

Definition 2.5. Let W be the set of sequences in $\mathbb{N}^{\mathbb{N}^+}$ having only finitely many non-zero entries. Let P be a finite set of iNtEgRaL polynomials. The **weight-vector** $w(P) \in W$ is the sequence where $w(P)_i$ is the number of distinct leading coefficients of polynomials in P of degree i .

We put a linear order on W , the **colex order**¹, as follows: $w < w'$ if $\exists i. w_i < w'_i \wedge \forall j > i. w_j = w'_j$. This is a well-order.

Lemma 2.6. Let P be a non-empty set of iNtEgRaL polynomials. Let $p_1(x) \in P$ be of minimal degree.

Let $P' := \{p(x) - p_1(x) : p \in P\}$. Then $w(P') < w(P)$.

Proof. If $\deg p > \deg p_1$, then $p - p_1$ has the same degree and the same leading coefficient.

If $\deg p = \deg p_1$ and they have distinct leading coefficients, then $\deg p - p_1 = \deg p$ and the number of distinct leading coefficients stays the same.

If $\deg p = \deg p_i$ and the leading coefficients agree, then the number of distinct leading coefficients decreases. □

Continuation of proof of Theorem 2.2 (Walters (98?)) We do outer induction on the weight vector and inner induction using color focussing.

Outer induction (on w using colex):

$\forall w \in W. \forall P$ finite set of iNtEgRaL polynomials $p_1(x), \dots, p_m(x)$

with $w(P) = w$.

$\forall k \geq 1. \exists N \geq 1. \forall k$ -coloring of $[N]. \exists a, d \neq 0$.

$\{a, a + p_1(d), \dots, a + p_m(d)\}$ is monochromatic.

Let P have weight vector w . Let $k \geq 1$. We can assume $p_j \neq 0$.

¹co-lexicographic

Missing picture

Inner induction (on r):

$$\forall 0 \leq r \leq k. \exists N. \forall k\text{-coloring of } [N].$$

- there exist $a, d \neq 0$ such that $a, \{a + p_j(d)\}_{i \leq j < m}$ lies in $[N]$ and is monochromatic or
- $\exists r$ color-focused tuples $\overline{b_1}, \dots, \overline{b_r}$ all in $[N]$ and focus a .

The case $r = 0$ is trivial. Let N satisfy the inner induction hypothesis for $r - 1$. We'll find N' for r .

Let d_{\max} be the maximal d such that there exists $a \in [N]$ with $\forall j. a + p_j(d) \in [N]$

Such a d_{\max} exists as for any polynomials $p(x)$, $\lim_{x \rightarrow \infty} p(x) = \pm \infty$.

Wlog. p has minimal degree in P . For $1 \leq j \leq m$, $0 \leq d \leq d_{\max}$, let $q_{j,d}(x) := \frac{1}{N} (p_j(Nx + d) - p_1(Nx) - p_j(d))$. Then each $q_{j,d}$ is an iNtEgRaL polynomial and the set $\{q_{j,d}(x)\}_{j,d}$ has a strictly smaller weight vector than P .

So using the outer induction hypothesis for N' large enough, we divide $[N']$ into blocks of size N and we are guaranteed identically colored blocks $B_s, B_{s+q_{j,d}(t)}$, where $1 \leq j \leq m$ and $0 \leq d \leq d_{\max}$. By the inner induction assumption, B_s contains $r - 1$ colour-focused tuples $\overline{b_1}, \dots, \overline{b_{r-1}}$ and their focus a , where a is of different color to $\overline{b_1}, \dots, \overline{b_{r-1}}$.

Write $\overline{b_i} = (a + p_j(d_i))_{1 \leq j \leq m}$.

Claim 2.2.1. $(a + Nq_{j,0}(t))_{1 \leq j \leq m}$,

$$\{(b_{i,j} + Nq_{j,d_i}(t))_j\}_i$$

are colour-focused at $a - p_1(Nt)$.

Subproof. Check the algebra. ■

□

Copy details from official notes

3 Galvin-Glazer

3rd talk, S. KAWAMOTO, 2024-04-22

Theorem 3.1. For every finite coloring of \mathbb{N} , there exists an infinite sequence such the finite sum of elements of the sequence are monochromatic.

For the proof we will construct a semigroup of ultrafilters $(\gamma\mathbb{N}, *)$ from $(\mathbb{N}, +)$ such that this is compact, Hausdorff, $*$ is associative and $\mathcal{U} \mapsto \mathcal{U} * \mathcal{V}$ continuous.

3.1 Semigroups, idempotents and ideals

Definition 3.2. A nonempty semigroup S with a compact and Hausdorff topology, such that $x \mapsto xs$ is continuous for all $s \in S$ is called a (right) **compact semigroup**.

Definition 3.3. An **idempotent** is an element $x \in S$ such that $x^2 = x$.

Lemma 3.4 (Ellis). Every compact semigroup contains an idempotent.

Proof. By Zorn's lemma, there exists a \subseteq -minimal compact sub-semigroup R .

Take $a \in R$. Then Ra is a compact semigroup and $Ra \subseteq R$, hence $Ra = R$. Define $P := \{x \in R : xa = a\}$. Since $Ra = R$, P is non-empty. Then P is a compact right semigroup, hence $P = R$ by the minimality of R . In particular $a \in P$, so $a^2 = a$. \square

Definition 3.5. A nonempty $I \subseteq S$ is called

- a **left-ideal** in S iff $SI \subseteq I$,
- a **right-ideal** in S iff $IS \subseteq I$,
- a **two-sided ideal** in S iff it is a left- and a right-ideal.

Observe. 1. For all $x \in S$, Sx is a closed left-ideal.

2. A minimal left-ideal is closed.

Definition 3.6. For $x, y \in S$, we let $x \leq y : \iff x = xy = yx$.

\leq is transitive and symmetric. Also $x \leq x$ iff x is an idempotent. So \leq defines a partial ordering on the set of idempotents in S .

Lemma 3.7. For an idempotent y and a closed left-ideal I , there exists $x \in Iy$ idempotent, such that $x \leq y$.

Proof. Iy is compact left-ideal. In particular, it is a right compact semigroup. So by Ellis' Lemma (3.4), there exists an idempotent $v = wy \in Iy$. Put $x := yv = ywy$. Then $x^2 = yw(yy)wy = y(wy)(wy) = ywy$, $xy = ywyy = ywy = x$ and $yx = yywy = ywy = x$. \square

Lemma 3.8. An idempotent is \leq -minimal iff it belongs to a minimal left-ideal.

Proof. “ \Leftarrow ” Let I be a minimal left-ideal and y be an idempotent. Given an idempotent x with $x \leq y$, we need to show $x = y$. Since $x \leq y$, $x = xy \in Iy = I$. Hence $y \in Ix = I$. Choose $z \in I$ such that $y = zx$. Then $x = yx = zxx = zx = y$.

“ \Rightarrow ” Let y be a minimal idempotent. Take an arbitrary minimal left-ideal I . By [Lemma 3.7](#), there exists an idempotent $x \in Iy$ such that $x \leq y$. But then $x = y$ by minimality of y and Iy is a minimal left-ideal. \square

Corollary 3.9. Any two-sided ideal contains all minimal left-ideals.

Proof. Let J be a two-sided ideal and y a minimal idempotent. Then by the lemma, there exists a minimal left-ideal containing y . We have $\emptyset \neq JI \subseteq I \cap J \subseteq I$, so by minimality of I , $J \cap I = I$, i.e. $I \subseteq J$. \square

Corollary 3.10. Let a be a minimal idempotent.

1. Sa is a minimal left-ideal.
2. aSa is a group.

Proof. 1. omitted.

2. $a \in aSa$, since a is an idempotent. Clearly a is an identity.

Left-inverse: Let $x = asa \in aSa$. We have $x \in aS$, so $Sx \subseteq Sa$. By 1. it follows that $Sa = Sx$. In particular, it follows that $a \in Sx$. Let $a = tx$. Put $y := ata$. We have $yx = atax = ataasa = atasa = atx = aa = a$.

The left-inverse is also a right-inverse.

\square

3.2 Ultrafilters

Definition 3.11. A nonempty set S with a partial map $\star: S^2 \rightarrow S$ with associativity is called a **partial semigroup**, i.e. whenever one side of the equation is defined, then both are defined.

An **adequate partial semigroup** (sometimes also **direct partial semigroup**) S satisfies

$$\forall n \in \mathbb{N}. \forall x_0, \dots, x_n \in S. \exists y \in S. y \neq x_i \text{ and } x_i \star y \text{ are all defined.}$$

Definition 3.12. Let (S, \star) be an adequate partial semigroup. We define

$$\gamma S := \{\text{Ultrafilter on } S : \forall x. (\mathcal{U}y)x \star y \text{ is defined}\}.$$

Consider the topology on γS generated by basic open sets of the form $\bar{A} := \{\mathcal{U} \in \gamma S : A \in \mathcal{U}\}$, $A \subseteq S$.

This turns γS into a compact Hausdorff-space.

Extend \star to γS , by letting

$$\mathcal{U} \star \mathcal{V} := \{A \subseteq S : (\mathcal{U}x)(\mathcal{V}y)x \star y \in A\}.$$

Fact 3.12.3. γS is closed under \star , $(\gamma S, \star)$ is associative and $\mathcal{U} \mapsto \mathcal{U} \star \mathcal{V}$ is continuous, i.e. $(\gamma S, \star)$ is a compact semigroup.

Corollary 3.13. If S is a compact right semigroup that has no idempotent or is left-cancellative, then $(\gamma S, \star)$ has a non-principal idempotent $\mathcal{U} \in \gamma S \setminus S$.

Definition 3.14. A (finite or infinite) sequence $\langle x_n \rangle \subseteq S$ of pairwise distinct x_i is called **basic** if for $n_0 < n_1 < \dots < n_k$, $x_{n_0} \star \dots \star x_{n_k}$ is defined.

Given a basic sequence $\langle x_n \rangle$, we define $[X] := \{x_{n_0} \star \dots \star x_{n_k}\}$.

Theorem 3.15 (Galvin-Glazer). Given an adequate partial semigroup S such what S has no idempotents or is left cancellative, for any finite coloring of S , we can find a basic sequence $\langle x_n \rangle$, such that $[X]$ is monochromatic.

Proof. We want to find $S \supseteq P_0 \supseteq P_1 \supseteq \dots$ and $x_n \in P_n$.

By **Corollary 3.13**, there is $\mathcal{U} \in \gamma S \setminus S$ idempotent.

Let $S = \bigsqcup_{i=1}^k S_i$ be the finite coloring. Choose i such that $S_i \in \mathcal{U}$ and set $P_0 := S_i$. Since $P_0 \in \mathcal{U} = \mathcal{U} \star \mathcal{U}$, we have $(\mathcal{U}y)(\mathcal{U}x)x \star y \in P_0$, so $\underbrace{\{y \in S : (\mathcal{U}x)x \star y \in P_0\}}_A \cap P_0 \in \mathcal{U}$. Pick $x_0 \in A \cap P_0$.

Let $P_1 := P_0 \cap \{y \in S : x_0 \star y \in P_0\}$. Again $P_1 \in \mathcal{U} = \mathcal{U} \star \mathcal{U}$ and we continue in a similar way.

We claim that $x_{n_0} \star \dots \star x_{n_k} \in P_{n_0}$. For $k = 0$, this is trivial. Assume now that the assertion holds up to $k - 1$. Then $x := x_{n_1} \star \dots \star x_{n_k} \in P_{n_1}$. By the definition of $P_{n_1} \subseteq P_{n_0+1}$, we get $x_{n_0} \star x \in P_{n_0}$.

□

4 Gowers' Theorem

4th talk, J. PIETSCH, 2024-04-29

The talk follows the presentation in [TT10, pp. 34–38].

Gowers' Theorem is a variant of the **Galvin-Glazer Theorem (3.15)** for the following partial semigroups, which have some additional structure:

Definition 4.1. For $k \geq 1$ let

$$\text{FIN}_k := \{f: \mathbb{N} \rightarrow \{0, \dots, k\} \mid \text{supp}(f) < \infty, k \in f[\mathbb{N}]\},$$

where

$$\text{supp}(f) := \{n \in \mathbb{N} \mid f(n) \neq 0\}.$$

Let $+$: $\text{FIN}_k \times \text{FIN}_k \rightarrow \text{FIN}_k$ denote the pointwise addition of **disjointly** supported elements. This turns $(\text{FIN}_k, +)$ into a directed partial semigroup.

The map

$$T: \text{FIN}_k \longrightarrow \text{FIN}_{k-1}$$

$$(f: \mathbb{N} \rightarrow \{0, \dots, k\}) \longmapsto \begin{pmatrix} \mathbb{N} & \longrightarrow & \{0, \dots, k-1\} \\ n & \longmapsto & \max(f(n) - 1, 0) \end{pmatrix}$$

is called the **tetris operation**.

Instead of basic sequences (cf. **Definition 3.14**) we consider so-called block sequences:

Definition 4.2. A **block sequence** is a (finite or infinite) sequence $\{b_n\}$ with $b_n \in \text{FIN}_k$ such that $\max(\text{supp}(b_n)) \leq \min(\text{supp}(b_{n+1}))$ for all i .

Given a block sequence $B = \{b_n\}$, the **partial subsemigroup generated by B** is defined to be

$$\langle B \rangle := \left\{ \sum_{i=1}^k T^{m_i}(b_{n_i}) \mid n_0 < n_1 < \dots < n_k, \exists i. m_i = 0 \right\}.$$

Now we can state the main theorem:

Theorem 4.3 (Gowers, 1992). For every finite coloring of FIN_k , there exists an infinite block sequence B , such that $\langle B \rangle$ is monochromatic.

For the proof we again use ultrafilters. As in the previous talk, we consider the semigroup $(\gamma \text{FIN}_k, +)$, where

$$\gamma \text{FIN}_k := \{\mathcal{U} \in \beta \text{FIN}_k : \forall x \in \text{FIN}_k. (\mathcal{U}x). x + y \text{ is defined}\}$$

and $+$ is extended to γFIN_k by letting

$$\mathcal{U} + \mathcal{V} := \{A \subseteq \text{FIN}_k : (\mathcal{U}x)(\mathcal{V}y). x + y \in A\}.$$

Note that in this setting, γFIN_k can be more explicitly described as

$$\gamma \text{FIN}_k = \{\mathcal{U} \in \beta \text{FIN}_k : \forall n \in \mathbb{N}. (\mathcal{U}x) \text{supp}(x) \cap \{0, \dots, n\} = \emptyset\}.$$

Thus γFIN_k is also called the set of **cofinite** ultrafilters.

We extend the tetris operation to $T: \gamma \text{FIN}_k \rightarrow \gamma \text{FIN}_{k-1}$, by

$$T(\mathcal{U}) := \{A \subseteq \text{FIN}_{k-1} : (\mathcal{U}x)T(x) \in A\}.$$

Lemma 4.4. $T: \gamma \text{FIN}_k \rightarrow \gamma \text{FIN}_{k-1}$ is a continuous, surjective homomorphism of semigroups.

Proof. Clearly T is surjective. Let $A \subseteq \text{FIN}_{k-1}$ and let $\bar{A} = \{\mathcal{U} \in \gamma \text{FIN}_{k-1} \mid A \in \mathcal{U}\}$ be the corresponding open subset. Then

$$\begin{aligned} T^{-1}[A] &= \{\mathcal{V} \in \gamma \text{FIN}_k \mid (\mathcal{V}x)T(x) \in A\} \\ &= \{x \in \text{FIN}_k \mid T(x) \in A\}. \end{aligned}$$

Furthermore

$$\begin{aligned} T(\mathcal{U} + \mathcal{V}) &= \{A \subseteq \text{FIN}_{k-1} : (\mathcal{U} + \mathcal{V}x). T(x) \in A\} \\ &= \{A \subseteq \text{FIN}_{k-1} : (\mathcal{U}x)(\mathcal{V}y). T(x + y) \in A\} \\ &= \{A \subseteq \text{FIN}_{k-1} : (\mathcal{U}x)(\mathcal{V}y). T(x) + T(y) \in A\} \\ &= \{A \subseteq \text{FIN}_{k-1} : (T(\mathcal{U})p)(T(\mathcal{V})q). p + q \in A\} \\ &= T(\mathcal{U}) + T(\mathcal{V}). \end{aligned}$$

□

As in the proof of the **Galvin-Glazer Theorem (3.15)** we want to use an idempotent ultrafilter. However in order to take care of the tetris operation, we need to construct a sequence of compatible idempotent ultrafilters as follows:

Lemma 4.5. There idempotent ultrafilters $\mathcal{U}_k \in \gamma \text{FIN}_k$, $k \in \mathbb{N}$ such that for all $j > i$:

- $\mathcal{U}_j \leq \mathcal{U}_i^a$ and
- $T^{j-i}(\mathcal{U}_j) = \mathcal{U}_i$.

^arecall that $\mathcal{U} \leq \mathcal{V} : \iff \mathcal{U} + \mathcal{V} = \mathcal{V} + \mathcal{U} = \mathcal{U}$

Proof. We construct the sequence recursively. We find \mathcal{U}_1 by applying [Ellis' Lemma \(3.4\)](#) to γFIN_1 . Suppose that $\mathcal{U}_1, \dots, \mathcal{U}_{k-1}$ have been chosen. We need to find a suitable \mathcal{U}_k . Let

$$S_k := \{\mathcal{X} \in \text{FIN}_k \mid T(\mathcal{X}) = \mathcal{U}_{k-1}\}.$$

Claim 1. $S_k + \mathcal{U}_{k-1}$ is a compact subsemigroup of γFIN_k .

Subproof. For $\mathcal{X}, \mathcal{Y} \in S_k$ we have

$$\begin{aligned} T(\mathcal{X} + \mathcal{U}_{k-1} + \mathcal{Y}) &\stackrel{\mathcal{X}, \mathcal{Y} \in S_k}{=} \mathcal{U}_k + \mathcal{U}_{k-1} + \mathcal{U}_k \\ &\stackrel{\mathcal{U}_k \leq \mathcal{U}_{k-1}}{=} \mathcal{U}_k, \end{aligned}$$

hence $(\mathcal{X} + \mathcal{U}_{k-1}) + (\mathcal{Y} + \mathcal{U}_{k-1}) \in S_k + \mathcal{U}_{k-1}$. S_k is compact, since T is continuous and $S_k + \mathcal{U}_{k-1}$ is compact, since $+$ is right-continuous. \blacksquare

By [Ellis' Lemma \(3.4\)](#) we find an idempotent $\mathcal{V} + \mathcal{U}_{k-1} \in S_k + \mathcal{U}_{k-1}$. Set $\mathcal{U}_k := \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1}$. Then

$$\begin{aligned} T(\mathcal{U}_k) &= T(\mathcal{U}_{k-1}) + T(\mathcal{V}) + T(\mathcal{U}_{k-2}) \\ &= \mathcal{U}_{k-2} + \mathcal{U}_{k-1} + \mathcal{U}_{k-2} \\ &\stackrel{\mathcal{U}_{k-1} \leq \mathcal{U}_{k-2}}{=} \mathcal{U}_{k-1}, \end{aligned}$$

\mathcal{U}_k is idempotent since

$$\begin{aligned} \mathcal{U}_k &= \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &\stackrel{\mathcal{V} + \mathcal{U}_{k-1} \text{ idempotent}}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &\stackrel{\mathcal{U}_{k-1} \text{ idempotent}}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &= \mathcal{U}_k + \mathcal{U}_k \end{aligned}$$

and for $l < k$ we have

$$\begin{aligned} \mathcal{U}_k + \mathcal{U}_l &= \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{U}_l \\ &\stackrel{\mathcal{U}_{k-1} \leq \mathcal{U}_l}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} = \mathcal{U}_k, \end{aligned}$$

hence $\mathcal{U}_k \leq \mathcal{U}_l$. \square

Proof of [Theorem 4.3](#). Fix $\mathcal{U}_n, n \leq k$ as in [Lemma 4.5](#).

Let A_0 be the piece of the partition such that $A_0 \in \mathcal{U}_k$. We define a sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ with $A_i \in \mathcal{U}_k$ and a block sequence $\{x_i\}$ such that

- (i) $\forall i. x_i \in A_i$ and

(ii) $\forall 1 \leq i, j \leq n. (\mathcal{U}_k x)[T^{k-i}(x_n) + T^{k-j}(x) \in A_n^{\max(i,j)}]$,

where $A_n^l := T^{k-l}[A_n]$.

We do this inductively: A suitable x_0 exists, since for all $i \leq j \leq k$

$$\begin{aligned} & (\mathcal{U}_k x). x \in A_0 \\ \implies & (T^{k-j}(\mathcal{U}_k x)). x \in A_0^j \\ \implies & (\mathcal{U}_j x) x \in A_0^j \\ \implies & ((\mathcal{U}_j + \mathcal{U}_i)x) x \in A_0^j \\ \implies & (\mathcal{U}_j x)(\mathcal{U}_i y) x + y \in A_0^j \\ \implies & (\mathcal{U}_k x)(\mathcal{U}_k y) T^{k-j}(x) + T^{k-i}(y) \in A_0^j \end{aligned}$$

and similarly for $j \leq i$, i.e. \mathcal{U} -almost all $x \in A_0$ work.

Suppose that A_0, \dots, A_n and x_0, \dots, x_n have been chosen. Let

$$C_n^{i,j} := \{x \in \text{FIN}_k \mid T^{k-i}(x_n) + T^{k-j}(x) \in A_n^{\max(i,j)}\}.$$

We have $C_n^{i,j} \in \mathcal{U}_k$ by (ii), so $A_{n+1} := A_n \cap \bigcap_{i,j} C_n^{i,j} \in \mathcal{U}_k$.

Note that

$$(\mathcal{U}_k x). \left((x \in A_{n+1}) \wedge (\mathcal{U}_k y). \forall 1 \leq i, j \leq k. T^{k-j}(x) + T^{k-i}(y) \in A_0^{\max(i,j)} \right),$$

so \mathcal{U}_k -almost all x can be chosen as x_{n+1} .

Claim 4.3.1. $\langle \{x_n\} \rangle$ is monochromatic.

Subproof. We show by induction on p , that

$$T^{k-l_0}(x_{n_0}) + \dots + T^{k-l_{p-1}}(x_{n_{p-1}}) + y \in A_{n_0}^{\max_i l_i}$$

for all $l_0, \dots, l_p \leq k$, $n_0 < n_2 < n_{p-1}$, $y \in A_{n_p}^{l_p}$.

The case of $p = 0$ follows from the condition (ii).

Suppose we have shown the statement for $p - 1$. Write

$$T^{k-l_0}(x_{n_0}) + \underbrace{T^{k-l_1}(x_{n_1}) + \dots + T^{k-l_{p-1}}(x_{n_{p-1}})}_{=: y'} + y.$$

By the induction assumption, we know that $y' \in A_{n_1}^l$, where $l := \max_{i>0} l_i$. Hence there exists $y^* \in A_{n_1}$ such that $y' = T^{k-l}(y^*)$. Since $y^* \in C_{n_0}^{l_0 l}$,

$$T^{k-l_0}(x_{n_0}) + T^{k-l}(y^*) \in A_{n_0}^{\max_i l_i}.$$

■

□

Corollary 4.6 (Hindman). For every finite coloring of FIN_1 , the set of finite subsets of \mathbb{N} , there exists an infinite block sequence B , such that $\langle B \rangle$, i.e. the set of unions of a finite, positive number of elements of B , is monochromatic.

Recall that c_0 is the Banach space of real sequences $(x_n)_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} x_n = 0$ and $\|(x_n)_{n \in \mathbb{N}}\| := \sup_{n \in \mathbb{N}} |x_n|$. Let $S_{c_0}^+ := \{x \in S_{c_0} \mid \|x\| = 1 \wedge \forall n. x_n \geq 0\}$ denote the positive part of the sphere of c_0 .

We can view FIN_k as a δ -net in $S_{c_0}^+$ in the following way:

Let δ be such that $\delta = \frac{1}{(1+\delta)^{k-1}}$. Define

$$\Phi_k: \text{FIN}_{[0,k]} \longrightarrow S_{c_0}^+$$

$$f \longmapsto \begin{pmatrix} \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & \begin{cases} \frac{1}{(1+\delta)^{k-f(n)}} & \text{if } f(n) > 0, \\ 0 & \text{otherwise,} \end{cases} \end{pmatrix}$$

where $\text{FIN}_{[0,k]} := \bigcup_{i=0}^k \text{FIN}_i$. Let $\Delta := \Phi_k[\text{FIN}_k]$. Note that the tetris operation corresponds to scalar multiplication in the sense that if $\lambda \in \mathbb{R}$ and $x \in \text{FIN}_k$ are such that $\lambda \Phi_k(x) \in \Phi_k[\text{FIN}_l]$ for some $l < k$, then $\lambda \Phi_k(x) = \Phi_k(T^{k-l}(x))$.

We obtain:

Corollary 4.7. For every $0 < \delta < 1$, there exists a δ -net Δ in $S_{c_0}^+$ such that for every finite coloring of Δ , there exists an infinite dimensional block subspace X of c_0 , such that $X \cap \Delta$ is monochromatic.

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