# Seminar: Infinite Ramsey Theory

Lecturer
Prof. Dr. Aleksandra Kwiatkowska

Notes
Josia Pietsch

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# Contents

These are my notes on the seminar "Infinite Ramsey Theory", taught by Prof. Dr. Aleksandra Kwiatkowska in the summer term 2024 at the University Münster.

**Warning 0.1.** This is not an official script. The official notes can be found here.

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# 1 A Couple of Proofs of the van der Waerden Theorem

$1^{\text{st}}$ talk, A. Kwiatkowska, 2024-04-08

TODO: Copy from official notes

# 2 The Polynomial van der Waerden Theorem

$2^{\text{nd}}$ talk, R. Sullivan, 2024-04-15

**Definition 2.1.** We say that $p \in \mathbb{Z}[x]$ is **iNtEgRaL**[a] iff $p(0) = 0$.

[a] in the paper, they were called **integral**, but this is clearly bad name.

Find a better name

**Theorem 2.2** (Polynomial van-der-Waerden)**.** Let $p_1(x), \ldots, p_m(x)$ be iNtEgRaL polynomials. Let $k \geqslant 1$. Then there exists $N \geqslant 1$ such that for any $k$-colouring of $[N]$, there exist $a, d \neq 0$, such that $a, a+p_1(d), \ldots, a+p_m(d)$ have the same color.

The classical van-der-Waerden can be recovered by setting $p_j(x) \coloneqq j \cdot x$. Classical van-der-Waerden is also called linear van-der-Waerden.

**Remark 2.2.1.** This is not true for $p(x)$ with $p(0) \neq 0:$ Let $p(x) = 2x+1$ and colour $\mathbb{N}_+$ by parity.

Originally, Theorem 2.2 was proved using ergodic theory. We will use similar techniques of the proof of classical van-der-Waerden. Recall: Color-focussing, double induction.

**Definition 2.3.** Let $P = \{p_1(x), \ldots, p_m(x)\}$ be a set of iNtEgRaL polynomials. We say that $\overline{b} \in \mathbb{Z}^m$ has **focus** $a \in \mathbb{Z}$ if $\exists d \neq 0.\ \forall 1 \leqslant j \leqslant m.\ b_j = a + p_j(d)$.

**Remark 2.3.2.** Note that this is the other way round to last week's proof.

For sets of polynomials $P$, it is not clear what "next" means.

**Definition 2.4.** Let $P = \{p_1(x), \dots, p_m(x)\}$. Fix a $k$-coloring of $[N]$. Let $\overline{b_1}, \dots, \overline{b_r} \in [N]^m$. We say $\overline{b_1}, \dots, \overline{b_r}$ are **color-focussed** at $a \in \mathbb{Z}$ if

- $a$ is a focus of each $\overline{b_i}$ and
- each $\overline{b_i}$ is monochromatic and the $\overline{b_i}$ are of different colours.

*Proof of Theorem 2.2 (Walters (98?))* We will first do the case of $p(x) = x^2$. For classical van-der-Waerden the strategy was to do

(i) an outer induction on the length of the arithmetic progression and

(ii) an inner induction using colour-focussing and blocks.

Since we only consider one polynomial, it suffices to do (ii).

We show, by induction on $r$ :

$$\forall 0 \leqslant r \leqslant k.\ \exists N.\ \forall \text{k}-\text{colourings of } [N] \tag{1}$$
$$\exists \text{ monochromatic } \{a, a + d^2\} \text{ or } r \text{ colour-focused } \{a + d_i^2\}_{0 \leqslant i \leqslant r} \text{ together with focus } a \tag{2}$$

The cases $r = 0, 1$ are trivial.

Let $r = 2$. Let $N$ witness Equation 2 for $n = 1$. We apply the same idea as in the linear van-der-Wareden. Use the induction hypothesis to get identically coloured blocks of size $N$ and find blocks correctly located so that we can do "jumps".

Missing picture, $f$ focus

We want $f, \lambda, \mu, q(x)$ such that

$$\begin{aligned} f + p(\lambda) &= a, \\ f + p(\mu) &= a + p(d) + Nq(t). \end{aligned}$$

So $q(t) = \frac{1}{N}(p(\mu) - p(\lambda))$. If this is iNtEgRaL and linear, we can use linear van-der-Waerden for this. So take $\mu = Nt + d$, $\lambda = Nt$,

$$q(t) = \frac{1}{N}\left(p(Nt + d) - p(Nt)\right) = 2dt.$$

Problem: $d$ depends on the particular colouring of the block $B$ and can't be fixed in advance. However, we know that $d$ is bounded ($d \leqslant N$). So use linear van-der-Waerden with the last term of the arithmetic progression $\leqslant s + 2Nt$. Once we look in $B_s$ and know $d$, we can fix $q(t) = 2dt$ and then find $B_{s+q(t)}$ coloured identically to $B_s$.

So we are now basically done: Given $N$ take $N'$ large enough using linear van-der-Waerden to guarantee that $B_s, B_{s+t}, \dots, B_{s+2Nt}$ coloured identically where each block is of size $N$.

There is one small issue: maybe $f = 0$. But there is an easy fix:

Everything we did was translation invariant. $f = a - p(Nt)$, $t \leqslant N'$. So take $N' + p(N \cdot N')$.

Missing picture

For general $r$, for each $a + d_i^2$ use $q_{d_i}(t) = 2d_i t$.

The only place where we used $p(x) = x^2$, was when applying linear van-der-Waerden.

In the general case we need to set up some induction. $\square$

**Definition 2.5.** Let $W$ be the set of sequences in $\mathbb{N}^{\mathbb{N}_+}$ having only finitely many non-zero entries. Let $P$ be a finite set of iNtEgRaL polynomials. The **weight-vector** $w(P) \in W$ is the sequence where $w(P)_i$ is the number of distinct leading coefficients of polynomials in $P$ of degree $i$.

We put a linear order on $W$, the **colex order**[1], as follows: $w < w'$ if $\exists i.\ w_i < w'_i \wedge \forall j > i.\ w_j = w'_j$. This is a well-order.

**Lemma 2.6.** Let $P$ be a non-empty set of iNtEgRaL polynomials. Let $p_1(x) \in P$ be of minimal degree.

Let $P' := \{p(x) - p_1(x) : p \in P\}$. Then $w(P') < w(P)$.

*Proof.* If $\deg p > \deg p_1$, then $p - p_1$ has the same degree and the same leading coefficient.

If $\deg p = \deg p_1$ and they have distinct leading coefficients, then $\deg p - p_1 = \deg p$ and the number of distinct leading coefficients stays the same.

If $\deg p = \deg p_i$ and the leading coefficients agree, then the number of distinct leading coefficients decreases. $\square$

*Continuation of proof of Theorem 2.2 (Walters (98?))* We do outer induction on the weight vector and inner induction using color focussing.

Outer induction (on $w$ unsing colex):

$$\begin{aligned}
&\forall w \in W.\ \forall P \text{ finite set of iNtEgRaL polynomials } p_1(x), \dots, p_m(x)\\
&\quad \text{with } w(P) = w.\\
&\ \forall k \geqslant 1.\ \exists N \geqslant 1.\ \forall k\text{-coloring of } [N].\ \exists a, d \neq 0.\\
&\ \{a, a + p_1(d), \dots, a + p_m(d)\} \text{ is monochromatic.}
\end{aligned}$$

Let $P$ have weight vector $w$. Let $k \geqslant 1$. We can assume $p_j \neq 0$.

[1] co-lexicographic

Inner induction (on $r$):

$$\forall 0 \leqslant r \leqslant k.\ \exists N.\ \forall k\text{-coloring of } [N].$$

- there exist $a, d \neq 0$ such that $a, \{a + p_j(d)\}_{i \leqslant j < m}$ lies in $[N]$ and is monochromatic or
- $\exists r$ color-focused tuples $\overline{b_1}, \ldots, \overline{b_r}$ all in $[N]$ and focus $a$.

The case $r = 0$ is trivial. Let $N$ satisfy the inner induction hypothesis for $r - 1$. We'll find $N'$ for $r$.

Let $d_{\max}$ be the maximal $d$ such that there exists $a \in [N]$ with $\forall j.\ a + p_j(d) \in [N]$

Such a $d_{\max}$ exists as for any polynomials $p(x)$, $\lim_{x\to\infty} p(x) = \pm\infty$.

Wlog. $p$ has minimal degree in $P$. For $1 \leqslant j \leqslant m$, $0 \leqslant d \leqslant d_{\max}$, let $q_{j,d}(x) := \frac{1}{N}\left(p_j(Nx + d) - p_1(Nx) - p_j(d)\right)$. Then each $q_{j,d}$ is an iNtEgRaL polynomial and the set $\{q_{j,d}(x)\}_{j,d}$ has a strictly smaller weight vector than $P$.

So using the outer induction hypothesis for $N'$ large enough, we divide $[N']$ into blocks of size $N$ and we are guaranteed identically colored blocks $B_s$, $B_{s+q_{j,d}(t)}$, where $1 \leqslant j \leqslant m$ and $0 \leqslant d \leqslant d_{\max}$. By the inner induction assumption, $B_s$ contains $r - 1$ colour-focused tuples $\overline{b_1}, \ldots, \overline{b_{r-1}}$ and their focus $a$, where $a$ is of different color to $\overline{b_1}, \ldots, \overline{b_{r-1}}$.

Write $\overline{b_i} = (a + p_j(d_i))_{1 \leqslant j \leqslant m}$.

**Claim 2.2.1.** $(a + Nq_{j,0}(t))_{1 \leqslant j \leqslant m}$,

$$\{(b_{i,j} + Nq_{j,d_i}(t))_j\}_i$$

*are colour-focused at $a - p_1(Nt)$.*

*Subproof.* Check the algebra. ■

Copy details from official notes

□

# 3 Galvin-Glazer

3rd talk, S. Kawamoto, 2024-04-22

**Theorem 3.1.** For every finite coloring of $\mathbb{N}$, there exists an infinite sequence such the finite sum of elements of the sequence are monochromatic.

For the proof we will construct a semigroup of ultrafilters $(\gamma\mathbb{N}, *)$ from $(\mathbb{N}, +)$ such that this is compact, Hausdorff, $*$ is associative and $\mathcal{U} \mapsto \mathcal{U} * \mathcal{V}$ continuous.

## 3.1 Semigroups, idempotents and ideals

**Definition 3.2.** A nonempty semigroup $S$ with a compact and Hausdorff topology, such that $x \mapsto xs$ is continuous for all $s \in S$ is called a (right) **compact semigroup**.

**Definition 3.3.** An **idempotent** is an element $x \in S$ such that $x^2 = x$.

**Lemma 3.4** (Ellis)**.** Every compact semigroup contains an idempotent.

*Proof.* By Zorn's lemma, there exists a $\subseteq$-minimal compact sub-semigroup $R$.

Take $a \in R$. Then $Ra$ is a compact semigroup and $Ra \subseteq R$, hence $Ra = R$. Define $P := \{x \in R : xa = a\}$. Since $Ra = R$, $P$ is non-empty. Then $P$ is a compact right semigroup, hence $P = R$ by the minimality of $R$. In particular $a \in P$, so $a^2 = a$. □

**Definition 3.5.** A nonempty $I \subseteq S$ is called

- a **left-ideal** in $S$ iff $SI \subseteq I$,
- a **right-ideal** in $S$ iff $IS \subseteq I$,
- a **two-sided ideal** in $S$ iff it is a left- and a right-ideal.

**Observe.** *1. For all $x \in S$, $Sx$ is a closed left-ideal.*

*2. A minimal left-ideal is closed.*

**Definition 3.6.** For $x, y \in S$, we let $x \leqslant y :\iff x = xy = yx$.

$\leqslant$ is transitive and symmetric. Also $x \leqslant x$ iff $x$ is an idempotent. So $\leqslant$ defines a partial ordering on the set of idempotents in $S$.

**Lemma 3.7.** For an idempotent $y$ and a closed left-ideal $I$, there exists $x \in Iy$ idempotent, such that $x \leqslant y$.

*Proof.* $Iy$ is compact left-ideal. In particular, it is a right compact semigroup. So by Ellis' Lemma (3.4), there exists an idempotent $v = wy \in Iy$. Put $x := yv = ywy$. Then $x^2 = yw(yy)wy = y(wy)(wy) = ywy$, $xy = ywyy = ywy = x$ and $yx = yywy = ywy = x$. □

**Lemma 3.8.** An idempotent is $\leqslant$-minimal iff it belongs to a minimal left-ideal.

*Proof.* "$\Longleftarrow$" Let $I$ be a minimal left-ideal and $y$ be an idempotent. Given an idempotent $x$ with $x \leqslant y$, we need to show $x = y$. Since $x \leqslant y$, $x = xy \in Iy = I$. Hence $y \in Ix = I$. Choose $z \in I$ such that $y = zx$. Then $x = yx = zxx = zx = y$.

"$\Longrightarrow$" Let $y$ be a minimal idempotent. Take an arbitrary minimal left-ideal $I$. By Lemma 3.7, there exists an idempotent $x \in Iy$ such that $x \leqslant y$. But then $x = y$ by minimality of $y$ and $Iy$ is a minimal left-ideal. ☐

**Corollary 3.9.** Any two-sided ideal contains all minimal left-ideals.

*Proof.* Let $J$ be a two-sides ideal and $y$ a minimal idempotent. Then by the lemma, there exists a minimal left-ideal containing $y$. We have $\varnothing \neq JI \subseteq I \cap J \subseteq I$, so by minimality of $I$, $J \cap I = I$, i.e. $I \subseteq J$. ☐

**Corollary 3.10.** Let $a$ be a minimal idempotent.

1. $Sa$ is a minimal left-ideal.
2. $aSa$ is a group.

*Proof.* 1. omitted.

2. $a \in aSa$, since $a$ is an idempotent. Clearly $a$ is an identity.

   Left-inverse: Let $x = asa \in aSa$. We have $x \in aS$, so $Sx \subseteq Sa$. By 1. it follows that $Sa = Sx$. In particular, it follows that $a \in Sx$. Let $a = tx$. Put $y := ata$. We have $yx = atax = ataasa = atasa = atx = aa = a$.

   The left-inverse is also a right-inverse.

☐

## 3.2 Ultrafilters

**Definition 3.11.** A nonempty set $S$ with a partial map $\star\colon S^2 \rightharpoonup S$ with associativity is called a **partial semigroup**, i.e. whenever one side of the equation is defined, then both are defined.

An **adequate partial semigroup** (somtimes also **direct partial semigroup**) $S$ satisfies

$$\forall n \in \mathbb{N}.\ \forall x_0, \ldots, x_n \in S.\ \exists y \in S.\ y \neq x_i \text{and } x_i \star y \text{ are all defined.}$$

**Definition 3.12.** Let $(S, \star)$ be an adequate partial semigroup. We define

$$\gamma S := \{\mathcal{U} \text{ultrafilter on } S : \forall x.\ (\mathcal{U} y) x \star y \text{ is defined}\}.$$

Consider the topology on $\gamma S$ generated by basic open sets of the form $\overline{A} := \{\mathcal{U} \in \gamma S : A \in \mathcal{U}\}$, $A \subseteq S$.

This turns $\gamma S$ into a compact Hausdorf-space.

Extend $\star$ to $\gamma S$, by letting

$$\mathcal{U} \star \mathcal{V} := \{A \subseteq S : (\mathcal{U}x)(\mathcal{V}y) x \star y \in A\}.$$

**Fact 3.12.3.** $\gamma S$ is closed under $\star$, $(\gamma S, \star)$ is associative and $\mathcal{U} \mapsto \mathcal{U} \star \mathcal{V}$ is continuous, i.e. $(\gamma S, \star)$ is a compact semigroup.

**Corollary 3.13.** If $S$ is a compact right semigroup that has no idempotent or is left-cancellative, then $(\gamma S, \star)$ has a non-principal idempotent $\mathcal{U} \in \gamma S \backslash S$.

**Definition 3.14.** A (finite or infinite) sequence $\langle x_n \rangle \subseteq S$ of pairwise distinct $x_i$ is called **basic** if for $n_0 < n_1 < \ldots < n_k$, $x_{n_0} \star .. \star x_{n_k}$ is defined.

Given a basic sequence $\langle x_n \rangle$, we define $[X] := \{x_{n_0} \star \ldots \star x_{n_k}\}$.

**Theorem 3.15** (Galvin-Glazer)**.** Given an adequate partial semigroup $S$ such what $S$ has no idempotents or is left cancellative, for any finite coloring of $S$, we can find a basic sequence $\langle x_n \rangle$, such that $[X]$ is monochromatic.

*Proof.* We want to find $S \supseteq P_0 \supseteq P_1 \supseteq \ldots$ and $x_n \in P_n$.

By Corollary 3.13, there is $\mathcal{U} \in \gamma S \backslash S$ idempotent.

Let $S = \bigsqcup_{i=1}^{k} S_i$ be the finite coloring. Choose $i$ such that $S_i \in \mathcal{U}$ and set $P_0 := S_i$. Since $P_0 \in \mathcal{U} = \mathcal{U} \star \mathcal{U}$, we have $(\mathcal{U}y)(\mathcal{U}x) x \star y \in P_0$, so $\underbrace{\{y \in S : (\mathcal{U}x) x \star y \in P_0\}}_{A} \cap P_0 \in \mathcal{U}$. Pick $x_0 \in A \cap P_0$.

Let $P_1 := P_0 \cap \{y \in S : x_0 \star y \in P_0\}$. Again $P_1 \in \mathcal{U} = \mathcal{U} \star \mathcal{U}$ and we continue in a similar way.

We claim that $x_{n_0} \star \ldots \star x_{n_k} \in P_{n_0}$. For $k = 0$, this is trivial. Assume now that the assertion holds up to $k-1$. Then $x := x_{n_1} \star \ldots \star x_{n_k} \in P_{n_1}$. By the definition of $P_{n_1} \subseteq P_{n_0+1}$, we get $x_{n_0} \star x \in P_{n_0}$.

□

# 4 Gowers' Theorem

$4^{\text{th}}$ talk, J. Pietsch, 2024-04-29

Gowers' Theorem is a variant of the Galvin-Glazer Theorem (3.15) for the following partial semigroups, which have some additional structure:

**Definition 4.1.** For $k \geqslant 1$ let

$$\mathrm{FIN}_k := \{f \colon \mathbb{N} \to \{0, \ldots, k\} \mid |\operatorname{supp}(f)| < \infty, k \in f[\mathbb{N}]\},$$

where

$$\operatorname{supp}(f) := \{n \in \mathbb{N} | f(n) \neq 0\}.$$

Let $+\colon \mathrm{FIN}_k \times \mathrm{FIN}_k \rightharpoonup \mathrm{FIN}_k$ denote the pointwise addition of **disjointly** supported elements. This turns $(\mathrm{FIN}_k, +)$ into a directed partial semigroup.

The map

$$\begin{aligned} T\colon \mathrm{FIN}_k &\longrightarrow \mathrm{FIN}_{k-1} \\ (f\colon \mathbb{N} \to \{0, \ldots, k\}) &\longmapsto \begin{pmatrix} \mathbb{N} & \longrightarrow & \{0, \ldots, k-1\} \\ n & \longmapsto & \max(f(n) - 1, 0) \end{pmatrix} \end{aligned}$$

is called the **tetris operation**.

Instead of basic sequences (cf. Definition 3.14) we consider so-called block sequences:

**Definition 4.2.** A **block sequence** is a (finite or infinite) sequence $\{b_n\}$ with $b_n \in \mathrm{FIN}_k$ such that $\max(\operatorname{supp}(b_n)) \leqslant \min(\operatorname{supp}_{b_{n+1}})$ for all $i$.

Given a block sequence $B = \{b_n\}$, the **partial subsemigroup generated by $B$** is defined to be

$$\langle B \rangle := \{\sum_{i=1}^{k} T^{m_i}(b_{n_i}) | n_0 < n_1 < \ldots n_k, \exists i.\ m_i = 0\}.$$

Now we can state the main theorem:

**Theorem 4.3** (Gowers, 1992)**.** For every finite coloring of $\mathrm{FIN}_k$, there exists an infinite block sequence $B$, such that $\langle B \rangle$ is monochromatic.

For the proof we again use ultrafilters. As in the previous talk, we consider the semigroup $(\gamma \mathrm{FIN}_k, +)$, where

$$\gamma \mathrm{FIN}_k := \{\mathcal{U} \in \beta \mathrm{FIN}_k : \forall x \in \mathrm{FIN}_k .\ (\mathcal{U}y).\ x + y \text{ is defined}\}$$

and $+$ is extended to $\gamma\,\mathrm{FIN}_k$ by letting

$$\mathcal{U}+\mathcal{V} := \{A \subseteq \mathrm{FIN}_k : (\mathcal{U}x)(\mathcal{V}y).\ x+y \in A\}.$$

Note that in this setting, $\gamma\,\mathrm{FIN}_k$ can be more explicitly described as

$$\gamma\,\mathrm{FIN}_k = \{\mathcal{U} \in \beta\,\mathrm{FIN}_k : \forall n \in \mathbb{N}.\ (\mathcal{U}x)\,\mathrm{supp}(x) \cap \{0,\dots,n\} = \varnothing\}.$$

Thus $\gamma\,\mathrm{FIN}_k$ is also called the set of **cofinite** ultrafilters.

We extend the tetris operation to $T\colon \gamma\,\mathrm{FIN}_k \to \gamma\,\mathrm{FIN}_{k-1}$, by

$$T(\mathcal{U}) := \{A \subseteq \mathrm{FIN}_{k-1} : (\mathcal{U}x)T(x) \in A\}.$$

**Lemma 4.4.** $T\colon \gamma\,\mathrm{FIN}_k \to \gamma\,\mathrm{FIN}_{k-1}$ is a continuous, surjective homomorphism of semigroups.

*Proof.* Clearly $T$ is surjective. Let $A \subseteq \mathrm{FIN}_{k-1}$ and let $\overline{A} = \{\mathcal{U} \in \gamma\,\mathrm{FIN}_{k-1} \,|\, A \in \mathcal{U}\}$ be the corresponding open subset. Then

$$\begin{aligned} T^{-1}[A] &= \{\mathcal{V} \in \gamma\,\mathrm{FIN}_k \,|(\mathcal{V}x)T(x) \in A\} \\ &= \overline{\{x \in \mathrm{FIN}_k \,|T(x) \in A\}}. \end{aligned}$$

Furthermore

$$\begin{aligned} T(\mathcal{U}+\mathcal{V}) &= \{A \subseteq \mathrm{FIN}_{k-1} : (\mathcal{U}+\mathcal{V}x).\ T(x) \in A\} \\ &= \{A \subseteq \mathrm{FIN}_{k-1} : (\mathcal{U}x)(\mathcal{V}y).\ T(x+y) \in A\} \\ &= \{A \subseteq \mathrm{FIN}_{k-1} : (\mathcal{U}x)(\mathcal{V}y).\ T(x)+T(y) \in A\} \\ &= \{A \subseteq \mathrm{FIN}_{k-1} : (T(\mathcal{U})p)(T(\mathcal{V})q).\ p+q \in A\} \\ &= T(\mathcal{U})+T(\mathcal{V}). \end{aligned}$$

□

As in the proof of the Galvin-Glazer Theorem (3.15) we want to use an idempotent ultrafilter. However in order to take care of the tetris operation, we need to construct a sequence of compatible idempotent ultrafilters as follows:

**Lemma 4.5.** There idempotent ultrafilters $\mathcal{U}_k \in \gamma\,\mathrm{FIN}_k$, $k \in \mathbb{N}$ such that for all $j > i$:

- $\mathcal{U}_j \leqslant \mathcal{U}_i$[a] and
- $T^{j-i}(\mathcal{U}_j) = \mathcal{U}_i$.

[a]recall that $\mathcal{U} \leqslant \mathcal{V} :\iff \mathcal{U}+\mathcal{V} = \mathcal{V}+\mathcal{U} = \mathcal{U}$

*Proof.* We construct the sequence recursively. We find $\mathcal{U}_1$ be applying Ellis' Lemma (3.4) to $\gamma\,\mathrm{FIN}_1$. Suppose that $\mathcal{U}_1, \ldots, \mathcal{U}_{k-1}$ have been chosen. We need to find a suitable $\mathcal{U}_k$. Let

$$S_k := \{\mathcal{X} \in \mathrm{FIN}_k \,|T(\mathcal{X}) = \mathcal{U}_{k-1}\}.$$

**Claim 1.** *$S_k + \mathcal{U}_{k-1}$ is a compact subsemigroup of $\gamma\,\mathrm{FIN}_k$.*

*Subproof.* For $\mathcal{X}, \mathcal{Y} \in S_k$ we have

$$\begin{aligned} T(\mathcal{X} + \mathcal{U}_{k-1} + \mathcal{Y}) &\overset{\mathcal{X},\mathcal{Y}\in S_k}{=} \mathcal{U}_k + \mathcal{U}_{k-1} + \mathcal{U}_k \\ &\overset{\mathcal{U}_k \leqslant \mathcal{U}_{k-1}}{=} \mathcal{U}_k, \end{aligned}$$

hence $(\mathcal{X} + \mathcal{U}_{k-1}) + (\mathcal{Y} + \mathcal{U}_{k-1}) \in S_k + \mathcal{U}_{k-1}$. $S_k$ is compact, since $T$ is continuous and $S_k + \mathcal{U}_{k-1}$ is compact, since $+$ is right-continuous. ■

By Ellis' Lemma (3.4) we find an idempotent $\mathcal{V} + \mathcal{U}_{k-1} \in S_k + \mathcal{U}_{k-1}$. Set $\mathcal{U}_k := \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1}$. Then

$$\begin{aligned} T(\mathcal{U}_k) &= T(\mathcal{U}_{k-1}) + T(\mathcal{V}) + T(\mathcal{U}_{k-2}) \\ &= \mathcal{U}_{k-2} + \mathcal{U}_{k-1} + \mathcal{U}_{k-2} \\ &\overset{\mathcal{U}_{k-1} \leqslant \mathcal{U}_{k-2}}{=} \mathcal{U}_{k-1}, \end{aligned}$$

$\mathcal{U}_k$ is idempotent since

$$\begin{aligned} \mathcal{U}_k &= \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &\overset{\mathcal{V}+\mathcal{U}_{k-1}\text{ idempotent}}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &\overset{\mathcal{U}_{k-1}\text{ idempotent}}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &= \mathcal{U}_k + \mathcal{U}_k \end{aligned}$$

and for $l < k$ we have

$$\begin{aligned} \mathcal{U}_k + \mathcal{U}_l &= \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{U}_l \\ &\overset{\mathcal{U}_{k-1} \leqslant \mathcal{U}_l}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} = \mathcal{U}_k, \end{aligned}$$

hence $\mathcal{U}_k \leqslant \mathcal{U}_l$. □

*Proof of Theorem 4.3.* Fix $\mathcal{U}_n, n \leqslant k$ as in Lemma 4.5.

Let $A_0$ be the piece of the partition such that $A_0 \in \mathcal{U}_k$. We define a sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ with $A_i \in \mathcal{U}_k$ and a block sequence $\{x_i\}$ such that

(i) $\forall i.\ x_i \in A_i$ and

(ii) $\forall 1 \leqslant i, j \leqslant n.\ (\mathcal{U}_k x)[T^{k-i}(x_n) + T^{k-j}(x) \in A_n^{\max(i,j)}]$,

where $A_n^l := T^{k-l}[A_n]$.

We do this inductively: A suitable $x_0$ exists, since for all $i \leqslant j \leqslant k$

$$\begin{aligned}
&(\mathcal{U}_k x).\ x \in A_0 \\
\implies &(T^{k-j}(\mathcal{U}_k)x).\ x \in A_0^j \\
\implies &(\mathcal{U}_j x)x \in A_0^j \\
\implies &((\mathcal{U}_j + \mathcal{U}_i)x)x \in A_0^j \\
\implies &(\mathcal{U}_j x)(\mathcal{U}_i y)x + y \in A_0^j \\
\implies &(\mathcal{U}_k x)(\mathcal{U}_k y)T^{k-j}(x) + T^{k-i}(y) \in A_0^j
\end{aligned}$$

and similarly for $j \leqslant i$, i.e. $\mathcal{U}$-almost all $x \in A_0$ work.

Suppose that $A_0, \ldots, A_n$ and $x_0, \ldots, x_n$ have been chosen. Let

$$C_n^{i,j} := \{x \in \mathrm{FIN}_k \,|T^{k-i}(x_n) + T^{k-j}(x) \in A_n^{\max(i,j)}\}.$$

We have $C_n^{i,j} \in \mathcal{U}_k$ by (ii), so $A_{n+1} := A_n \cap \bigcap_{i,j} C_n^{i,j} \in \mathcal{U}_k$.

Note that

$$(\mathcal{U}_k x).\ \left((x \in A_{n+1}) \wedge (\mathcal{U}_k y).\ \forall 1 \leqslant i, j \leqslant k.\ T^{k-j}(x) + T^{k-i}(y) \in A_0^{\max(i,j)}\right),$$

so $\mathcal{U}_k$-almost all $x$ can be chosen as $x_{n+1}$.

**Claim 4.3.1.** *$\langle\{x_n\}\rangle$ is monochromatic.*

*Subproof.* We show by induction on $p$, that

$$T^{k-l_0}(x_{n_0}) + \ldots + T^{k-l_{p-1}}(x_{n_{p-1}}) + y \in A_{n_0}^{\max_i l_i}$$

for all $l_0, \ldots, l_p \leqslant k$, $n_0 < n_2 < n_{p-1}$, $y \in A_{n_p}^{l_p}$.

The case of $p = 0$ follows from the condition (ii).

Suppose we have shown the statement for $p - 1$. Write

$$T^{k-l_0}(x_{n_0}) + \underbrace{T^{k-l_1}(x_{n_1}) + \ldots + T^{k-l_{p-1}}(x_{n_{p-1}}) + y}_{=:y'}.$$

By the induction assumption, we know that $y' \in A_{n_1}^l$, where $l := \max_{i>0} l_i$. Hence there exists $y^* \in A_{n_1}$ such that $y' = T^{k-l}(y^*)$. Since $y^* \in C_{n_0}^{l_0 l}$,

$$T^{k-l_0}(x_{n_0}) + T^{k-l}(y^*) \in A_{n_0}^{\max l_i}.$$

■

□

**Corollary 4.6** (Hindman)**.** For every finite coloring of $\mathrm{FIN}_1$, the set of finite subsets of $\mathbb{N}$, there exists an infinite block sequence $B$, such that $\langle B \rangle$, i.e. the set of unions of a finite, positive number of elements of $B$, is monochromatic.

Recall that $c_0$ is the Banach space of real sequences $(x_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} x_n = 0$ and $\|(x_n)_{n\in\mathbb{N}}\| := \sup_{n\in\mathbb{N}} |x_n|$. Let $S^+_{c_0} := \{x \in S_{c_0} \,|\, \|x\| = 1 \wedge \forall n.\ x_n \geqslant 0\}$ denote the positive part of the sphere of $c_0$.

We can view $\mathrm{FIN}_k$ as a $\delta$-net in $S^+_{c_0}$ in the following way:

Let $\delta$ be such that $\delta = \frac{1}{(1+\delta)^{k-1}}$. Define

$$\begin{aligned} \Phi_k \colon\ \mathrm{FIN}_{[0,k]} &\longrightarrow S^+_{c_0} \\ f &\longmapsto \begin{pmatrix} \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & \begin{cases} \frac{1}{(1+\delta)^{k-f(n)}} & \text{if } f(n) > 0, \\ 0 & \text{otherwise,} \end{cases} \end{pmatrix} \end{aligned}$$

where $\mathrm{FIN}_{[0,k]} := \bigcup_{i=0}^{k} \mathrm{FIN}_k$. Let $\Delta := \Phi_k[\mathrm{FIN}_k]$. Note that the tetris operation corresponds to scalar multiplication in the sense that if $\lambda \in \mathbb{R}$ and $x \in \mathrm{FIN}_k$ are such that $\lambda\Phi_k(x) \in \Phi_k[\mathrm{FIN}_l]$ for some $l < k$, then $\lambda\Phi_k(x) = \Phi_k(T^{k-l}(x))$.

We obtain:

**Corollary 4.7.** For every $0 < \delta < 1$, there exists a $\delta$-net $\Delta$ in $S^+_{c_0}$ such that for every finite coloring of $\Delta$, there exists an infinite dimensional block subspace $X$ of $c_0$, such that $X \cap \Delta$ is monochromatic.

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