Seminar: Infinite Ramsey Theory

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These are my notes on the seminar "Infinite Ramsey Theory", taught by Prof. Dr. Aleksandra Kwiatkowska in the summer term 2024 at the University Münster.

Warning 0.1. This is not an official script. The official notes can be found here.

I wrote these notes while attending the seminar. There are a lot of problems and unfortunately I currently do not have time to clean this up. Section 4 serves as the official notes for my own talk. Other than that this document is largely incomplete, especially the notes on the later, more technical talks. So it's probably a good idea to read the resp. official notes of the talks.

If you find errors or want to improve something, please send me a message: `lecturenotes@jrpie.de`.

1 A Couple of Proofs of the van der Waerden Theorem

1st talk, A. Kwiatkowska, 2024-04-08

Unfortunately I didn't attend this talk. However there are official notes.

2 The Polynomial van der Waerden Theorem

2nd talk, R. Sullivan, 2024-04-15

There are official notes.

Definition 2.1. We say that $p \in \mathbb{Z}[x]$ is **iNtEgRaL**[a] iff $p(0) = 0$.

Find a better name

[a] in the paper, they were called **integral**, but this is clearly bad name.

Theorem 2.2 (Polynomial van-der-Waerden)**.** Let $p_1(x), \dots, p_m(x)$ be iNtEgRaL polynomials. Let $k \geqslant 1$. Then there exists $N \geqslant 1$ such that for any k-colouring of $[N]$, there exist $a, d \neq 0$, such that $a, a + p_1(d), \dots, a + p_m(d)$ have the same color.

The classical van-der-Waerden can be recovered by setting $p_j(x) := j \cdot x$. Classical van-der-Waerden is also called linear van-der-Waerden.

Remark 2.2.1. This is not true for $p(x)$ with $p(0) \neq 0$: Let $p(x) = 2x + 1$ and colour $\mathbb{N}_+$ by parity.

Originally, Theorem 2.2 was proved using ergodic theory. We will use similar techniques of the proof of classical van-der-Waerden. Recall: Color-focussing, double induction.

Definition 2.3. Let $P = \{p_1(x), \ldots, p_m(x)\}$ be a set of iNtEgRaL polynomials. We say that $\overline{b} \in \mathbb{Z}^m$ has **focus** $a \in \mathbb{Z}$ if $\exists d \neq 0.\ \forall 1 \leqslant j \leqslant m.\ b_j = a + p_j(d)$.

Remark 2.3.2. Note that this is the other way round to last week's proof.

For sets of polynomials P, it is not clear what "next" means.

Definition 2.4. Let $P = \{p_1(x), \ldots, p_m(x)\}$. Fix a k-coloring of $[N]$. Let $\overline{b_1}, \ldots, \overline{b_r} \in [N]^m$. We say $\overline{b_1}, \ldots, \overline{b_r}$ are **color-focussed** at $a \in \mathbb{Z}$ if

- a is a focus of each $\overline{b_i}$ and
- each $\overline{b_i}$ is monochromatic and the $\overline{b_i}$ are of different colours.

Proof of Theorem 2.2 (Walters (98?)) We will first do the case of $p(x) = x^2$. For classical van-der-Waerden the strategy was to do

(i) an outer induction on the length of the arithmetic progression and

(ii) an inner induction using colour-focussing and blocks.

Since we only consider one polynomial, it suffices to do (ii).

We show, by induction on r :

$$\forall 0 \leqslant r \leqslant k.\ \exists N.\ \forall \text{k}-\text{colourings of } [N] \tag{1}$$

$$\exists \text{ monochromatic } \{a, a + d^2\} \text{ or } r \text{ colour-focused } \{a + d_i^2\}_{0 \leqslant i \leqslant r} \text{ together with focus } a \tag{2}$$

The cases $r = 0, 1$ are trivial.

Let $r = 2$. Let N witness Equation 2 for $n = 1$. We apply the same idea as in the linear van-der-Wareden. Use the induction hypothesis to get identically coloured blocks of size N and find blocks correctly located so that we can do "jumps".

Missing picture, f focus

We want $f, \lambda, \mu, q(x)$ such that

$$\begin{aligned} f + p(\lambda) &= a, \\ f + p(\mu) &= a + p(d) + Nq(t). \end{aligned}$$

So $q(t) = \frac{1}{N}(p(\mu) - p(\lambda))$. If this is iNtEgRaL and linear, we can use linear van-der-Waerden for this. So take $\mu = Nt + d$, $\lambda = Nt$,

$$q(t) = \frac{1}{N}\left(p(Nt + d) - p(Nt)\right) = 2dt.$$

Problem: d depends on the particular colouring of the block B and can't be fixed in advance. However, we know that d is bounded ($d \leqslant N$). So use linear

van-der-Waerden with the last term of the arithmetic progression $\leqslant s + 2Nt$. Once we look in B_s and know d, we can fix $q(t) = 2dt$ and then find $B_{s+q(t)}$ coloured identically to B_s.

So we are now basically done: Given N take N' large enough using linear van-der-Waerden to guarantee that $B_s, B_{s+t}, \ldots, B_{s+2Nt}$ coloured identically where each block is of size N.

There is one small issue: maybe $f = 0$. But there is an easy fix:

Everything we did was translation invariant. $f = a - p(Nt)$, $t \leqslant N'$. So take $N' + p(N \cdot N')$.

Missing picture

For general r, for each $a + d_i^2$ use $q_{d_i}(t) = 2d_i t$.

The only place where we used $p(x) = x^2$, was when applying linear van-der-Waerden.

In the general case we need to set up some induction. □

Definition 2.5. Let W be the set of sequences in $\mathbb{N}^{\mathbb{N}_+}$ having only finitely many non-zero entries. Let P be a finite set of iNtEgRaL polynomials. The **weight-vector** $w(P) \in W$ is the sequence where $w(P)_i$ is the number of distinct leading coefficients of polynomials in P of degree i.

We put a linear order on W, the **colex order**[1], as follows: $w < w'$ if $\exists i.\ w_i < w'_i \wedge \forall j > i.\ w_j = w'_j$. This is a well-order.

Lemma 2.6. Let P be a non-empty set of iNtEgRaL polynomials. Let $p_1(x) \in P$ be of minimal degree.

Let $P' := \{p(x) - p_1(x) : p \in P\}$. Then $w(P') < w(P)$.

Proof. If $\deg p > \deg p_1$, then $p - p_1$ has the same degree and the same leading coefficient.

If $\deg p = \deg p_1$ and they have distinct leading coefficients, then $\deg p - p_1 = \deg p$ and the number of distinct leading coefficients stays the same.

If $\deg p = \deg p_i$ and the leading coefficients agree, then the number of distinct leading coefficients decreases. □

Continuation of proof of Theorem 2.2 (Walters (98?)) We do outer induction on the weight vector and inner induction using color focussing.

[1] co-lexicographic

Outer induction (on w unsing colex):

$$\begin{aligned}&\forall w \in W.\ \forall P \text{ finite set of iNtEgRaL polynomials } p_1(x), \ldots, p_m(x)\\ &\quad \text{with } w(P) = w.\\ &\forall k \geqslant 1.\ \exists N \geqslant 1.\ \forall k\text{-coloring of } [N].\ \exists a, d \neq 0.\\ &\{a, a + p_1(d), \ldots, a + p_m(d)\} \text{ is monochromatic.}\end{aligned}$$

Let P have weight vector w. Let $k \geqslant 1$. We can assume $p_j \neq 0$.

Inner induction (on r):

$$\forall 0 \leqslant r \leqslant k.\ \exists N.\ \forall k\text{-coloring of } [N].$$

- there exist $a, d \neq 0$ such that $a, \{a + p_j(d)\}_{i \leqslant j < m}$ lies in $[N]$ and is monochromatic or
- $\exists r$ color-focused tuples $\overline{b_1}, \ldots, \overline{b_r}$ all in $[N]$ and focus a.

The case $r = 0$ is trivial. Let N satisfy the inner induction hypothesis for $r - 1$. We'll find N' for r.

Let $d_{\max}$ be the maximal d such that there exists $a \in [N]$ with $\forall j.\ a + p_j(d) \in [N]$

Such a $d_{\max}$ exists as for any polynomials $p(x)$, $\lim_{x \to \infty} p(x) = \pm\infty$.

Wlog. p has minimal degree in P. For $1 \leqslant j \leqslant m$, $0 \leqslant d \leqslant d_{\max}$, let $q_{j,d}(x) := \frac{1}{N}\left(p_j(Nx + d) - p_1(Nx) - p_j(d)\right)$. Then each $q_{j,d}$ is an iNtEgRaL polynomial and the set $\{q_{j,d}(x)\}_{j,d}$ has a strictly smaller weight vector than P.

So using the outer induction hypothesis for N' large enough, we divide $[N']$ into blocks of size N and we are guaranteed identically colored blocks B_s, $B_{s+q_{j,d}(t)}$, where $1 \leqslant j \leqslant m$ and $0 \leqslant d \leqslant d_{\max}$. By the inner induction assumption, B_s contains $r - 1$ colour-focused tuples $\overline{b_1}, \ldots, \overline{b_{r-1}}$ and their focus a, where a is of different color to $\overline{b_1}, \ldots, \overline{b_{r-1}}$.

Write $\overline{b_i} = (a + p_j(d_i))_{1 \leqslant j \leqslant m}$.

Claim 2.2.1. $(a + Nq_{j,0}(t))_{1 \leqslant j \leqslant m}$,

$$\{(b_{i,j} + Nq_{j,d_i}(t))_j\}_i$$

are colour-focused at $a - p_1(Nt)$.

Subproof. Check the algebra. ■

Copy details from official notes

□

3 Galvin-Glazer

3rd talk, S. Kawamoto, 2024-04-22

Link to the official notes.

Theorem 3.1. For every finite coloring of $\mathbb{N}$, there exists an infinite sequence such the finite sum of elements of the sequence are monochromatic.

For the proof we will construct a semigroup of ultrafilters $(\gamma\mathbb{N}, *)$ from $(\mathbb{N}, +)$ such that this is compact, Hausdorff, $*$ is associative and $\mathcal{U} \mapsto \mathcal{U} * \mathcal{V}$ continuous.

3.1 Semigroups, idempotents and ideals

Definition 3.2. A nonempty semigroup S with a compact and Hausdorff topology, such that $x \mapsto xs$ is continuous for all $s \in S$ is called a (right) **compact semigroup**.

Definition 3.3. An **idempotent** is an element $x \in S$ such that $x^2 = x$.

Lemma 3.4 (Ellis)**.** Every compact semigroup contains an idempotent.

Proof. By Zorn's lemma, there exists a $\subseteq$-minimal compact sub-semigroup R.

Take $a \in R$. Then Ra is a compact semigroup and $Ra \subseteq R$, hence $Ra = R$. Define $P := \{x \in R : xa = a\}$. Since $Ra = R$, P is non-empty. Then P is a compact right semigroup, hence $P = R$ by the minimality of R. In particular $a \in P$, so $a^2 = a$. $\square$

Definition 3.5. A nonempty $I \subseteq S$ is called

- a **left-ideal** in S iff $SI \subseteq I$,
- a **right-ideal** in S iff $IS \subseteq I$,
- a **two-sided ideal** in S iff it is a left- and a right-ideal.

Observe. *1. For all $x \in S$, Sx is a closed left-ideal.*

2. A minimal left-ideal is closed.

Definition 3.6. For $x, y \in S$, we let $x \leqslant y :\iff x = xy = yx$.

$\leqslant$ is transitive and symmetric. Also $x \leqslant x$ iff x is an idempotent. So $\leqslant$ defines a partial ordering on the set of idempotents in S.

Lemma 3.7. For an idempotent y and a closed left-ideal I, there exists

$x \in Iy$ idempotent, such that $x \leqslant y$.

Proof. Iy is compact left-ideal. In particular, it is a right compact semigroup. So by Ellis' Lemma (3.4), there exists an idempotent $v = wy \in Iy$. Put $x := yv = ywy$. Then $x^2 = yw(yy)wy = y(wy)(wy) = ywy$, $xy = ywyy = ywy = x$ and $yx = yywy = ywy = x$. □

Lemma 3.8. An idempotent is $\leqslant$-minimal iff it belongs to a minimal left-ideal.

Proof. "$\Longleftarrow$" Let I be a minimal left-ideal and y be an idempotent. Given an idempotent x with $x \leqslant y$, we need to show $x = y$. Since $x \leqslant y$, $x = xy \in Iy = I$. Hence $y \in Ix = I$. Choose $z \in I$ such that $y = zx$. Then $x = yx = zxx = zx = y$.

"$\Longrightarrow$" Let y be a minimal idempotent. Take an arbitrary minimal left-ideal I. By Lemma 3.7, there exists an idempotent $x \in Iy$ such that $x \leqslant y$. But then $x = y$ by minimality of y and Iy is a minimal left-ideal. □

Corollary 3.9. Any two-sided ideal contains all minimal left-ideals.

Proof. Let J be a two-sides ideal and y a minimal idempotent. Then by the lemma, there exists a minimal left-ideal containing y. We have $\varnothing \neq JI \subseteq I \cap J \subseteq I$, so by minimality of I, $J \cap I = I$, i.e. $I \subseteq J$. □

Corollary 3.10. Let a be a minimal idempotent.

1. Sa is a minimal left-ideal.
2. aSa is a group.

Proof. 1. omitted.

2. $a \in aSa$, since a is an idempotent. Clearly a is an identity.

   Left-inverse: Let $x = asa \in aSa$. We have $x \in aS$, so $Sx \subseteq Sa$. By 1. it follows that $Sa = Sx$. In particular, it follows that $a \in Sx$. Let $a = tx$. Put $y := ata$. We have $yx = atax = ataasa = atasa = atx = aa = a$.

   The left-inverse is also a right-inverse.

□

3.2 Ultrafilters

Definition 3.11. A nonempty set S with a partial map $\star\colon S^2 \rightharpoonup S$ with

associativity is called a **partial semigroup**, i.e. whenever one side of the equation is defined, then both are defined.

An **adequate partial semigroup** (somtimes also **direct partial semigroup**) S satisfies

$$\forall n \in \mathbb{N}.\ \forall x_0, \ldots, x_n \in S.\ \exists y \in S.\ y \neq x_i \text{and } x_i \star y \text{ are all defined.}$$

Definition 3.12. Let $(S, \star)$ be an adequate partial semigroup. We define

$$\gamma S := \{\mathcal{U}\text{ultrafilter on } S : \forall x.\ (\mathcal{U}y) x \star y \text{ is defined}\}.$$

Consider the topology on γS generated by basic open sets of the form $\overline{A} := \{\mathcal{U} \in \gamma S : A \in \mathcal{U}\}$, $A \subseteq S$.

This turns γS into a compact Hausdorf-space.

Extend $\star$ to γS, by letting

$$\mathcal{U} \star \mathcal{V} := \{A \subseteq S : (\mathcal{U}x)(\mathcal{V}y) x \star y \in A\}.$$

Fact 3.12.3. γS is closed under $\star$, $(\gamma S, \star)$ is associative and $\mathcal{U} \mapsto \mathcal{U} \star \mathcal{V}$ is continuous, i.e. $(\gamma S, \star)$ is a compact semigroup.

Corollary 3.13. If S is a compact right semigroup that has no idempotent or is left-cancellative, then $(\gamma S, \star)$ has a non-principal idempotent $\mathcal{U} \in \gamma S \backslash S$.

Definition 3.14. A (finite or infinite) sequence $\langle x_n \rangle \subseteq S$ of pairwise distinct x_i is called **basic** if for $n_0 < n_1 < \ldots < n_k$, $x_{n_0} \star .. \star x_{n_k}$ is defined.

Given a basic sequence $\langle x_n \rangle$, we define $[X] := \{x_{n_0} \star \ldots \star x_{n_k}\}$.

Theorem 3.15 (Galvin-Glazer)**.** Given an adequate partial semigroup S such what S has no idempotents or is left cancellative, for any finite coloring of S, we can find a basic sequence $\langle x_n \rangle$, such that $[X]$ is monochromatic.

Proof. We want to find $S \supseteq P_0 \supseteq P_1 \supseteq \ldots$ and $x_n \in P_n$.

By Corollary 3.13, there is $\mathcal{U} \in \gamma S \backslash S$ idempotent.

Let $S = \bigsqcup_{i=1}^{k} S_i$ be the finite coloring. Choose i such that $S_i \in \mathcal{U}$ and set $P_0 := S_i$. Since $P_0 \in \mathcal{U} = \mathcal{U} \star \mathcal{U}$, we have $(\mathcal{U}y)(\mathcal{U}x) x \star y \in P_0$, so $\underbrace{\{y \in S : (\mathcal{U}x) x \star y \in P_0\}}_{A} \cap P_0 \in \mathcal{U}$. Pick $x_0 \in A \cap P_0$.

Let $P_1 \coloneqq P_0 \cap \{y \in S : x_0 \star y \in P_0\}$. Again $P_1 \in \mathcal{U} = \mathcal{U} \star \mathcal{U}$ and we continue in a similar way.

We claim that $x_{n_0} \star \ldots \star x_{n_k} \in P_{n_0}$. For $k = 0$, this is trivial. Assume now that the assertion holds up to $k-1$. Then $x \coloneqq x_{n_1} \star \ldots \star x_{n_k} \in P_{n_1}$. By the definition of $P_{n_1} \subseteq P_{n_0+1}$, we get $x_{n_0} \star x \in P_{n_0}$.

$\square$

4 Gowers' Theorem

4th talk, J. Pietsch, 2024-04-29

The talk follows the presentation in [TT10, pp. 34–38].

Gowers' Theorem is a variant of the Galvin-Glazer Theorem (3.15) for the following partial semigroups, which have some additional structure:

Definition 4.1. For $k \geqslant 1$ let

$$\mathrm{FIN}_k := \{f\colon \mathbb{N} \to \{0,\dots,k\} \mid |\operatorname{supp}(f)| < \infty, k \in f[\mathbb{N}]\},$$

where

$$\operatorname{supp}(f) := \{n \in \mathbb{N} \mid f(n) \neq 0\}.$$

Let $+\colon \mathrm{FIN}_k \times \mathrm{FIN}_k \rightharpoonup \mathrm{FIN}_k$ denote the pointwise addition of **disjointly** supported elements. This turns $(\mathrm{FIN}_k, +)$ into a directed partial semigroup.

The map

$$\begin{aligned} T\colon \mathrm{FIN}_k &\longrightarrow \mathrm{FIN}_{k-1} \\ (f\colon \mathbb{N} \to \{0,\dots,k\}) &\longmapsto \begin{pmatrix} \mathbb{N} & \longrightarrow & \{0,\dots,k-1\} \\ n & \longmapsto & \max(f(n)-1, 0) \end{pmatrix} \end{aligned}$$

is called the **tetris operation**.

Instead of basic sequences (cf. Definition 3.14) we consider so-called block sequences:

Definition 4.2. A **block sequence** is a (finite or infinite) sequence $\{b_n\}$ with $b_n \in \mathrm{FIN}_k$ such that $\max(\operatorname{supp}(b_n)) \leqslant \min(\operatorname{supp}_{b_{n+1}})$ for all i.

Given a block sequence $B = \{b_n\}$, the **partial subsemigroup generated by B** is defined to be

$$\langle B \rangle := \{\sum_{i=1}^{k} T^{m_i}(b_{n_i}) \mid n_0 < n_1 < \dots n_k, \exists i.\ m_i = 0\}.$$

Now we can state the main theorem:

Theorem 4.3 (Gowers, 1992)**.** For every finite coloring of FIN_k, there exists an infinite block sequence B, such that $\langle B \rangle$ is monochromatic.

For the proof we again use ultrafilters. As in the previous talk, we consider the semigroup $(\gamma\,\mathrm{FIN}_k, +)$, where

$$\gamma\,\mathrm{FIN}_k := \{\mathcal{U} \in \beta\,\mathrm{FIN}_k : \forall x \in \mathrm{FIN}_k\,.\ (\mathcal{U}y).\ x + y \text{ is defined}\}$$

and $+$ is extended to $\gamma\,\mathrm{FIN}_k$ by letting

$$\mathcal{U} + \mathcal{V} := \{A \subseteq \mathrm{FIN}_k : (\mathcal{U}x)(\mathcal{V}y).\ x + y \in A\}.$$

Note that in this setting, $\gamma\,\mathrm{FIN}_k$ can be more explicitly described as

$$\gamma\,\mathrm{FIN}_k = \{\mathcal{U} \in \beta\,\mathrm{FIN}_k : \forall n \in \mathbb{N}.\ (\mathcal{U}x)\,\mathrm{supp}(x) \cap \{0, \dots, n\} = \varnothing\}.$$

Thus $\gamma\,\mathrm{FIN}_k$ is also called the set of **cofinite** ultrafilters.

We extend the tetris operation to $T\colon \gamma\,\mathrm{FIN}_k \to \gamma\,\mathrm{FIN}_{k-1}$, by

$$T(\mathcal{U}) := \{A \subseteq \mathrm{FIN}_{k-1} : (\mathcal{U}x)T(x) \in A\}.$$

Lemma 4.4. $T\colon \gamma\,\mathrm{FIN}_k \to \gamma\,\mathrm{FIN}_{k-1}$ is a continuous, surjective homomorphism of semigroups.

Proof. Clearly T is surjective. Let $A \subseteq \mathrm{FIN}_{k-1}$ and let $\overline{A} = \{\mathcal{U} \in \gamma\,\mathrm{FIN}_{k-1} \,|A \in \mathcal{U}\}$ be the corresponding open subset. Then

$$\begin{aligned} T^{-1}[A] &= \{\mathcal{V} \in \gamma\,\mathrm{FIN}_k \,|(\mathcal{V}x)T(x) \in A\} \\ &= \overline{\{x \in \mathrm{FIN}_k \,|T(x) \in A\}}. \end{aligned}$$

Furthermore

$$\begin{aligned} T(\mathcal{U} + \mathcal{V}) &= \{A \subseteq \mathrm{FIN}_{k-1} : (\mathcal{U} + \mathcal{V}x).\ T(x) \in A\} \\ &= \{A \subseteq \mathrm{FIN}_{k-1} : (\mathcal{U}x)(\mathcal{V}y).\ T(x + y) \in A\} \\ &= \{A \subseteq \mathrm{FIN}_{k-1} : (\mathcal{U}x)(\mathcal{V}y).\ T(x) + T(y) \in A\} \\ &= \{A \subseteq \mathrm{FIN}_{k-1} : (T(\mathcal{U})p)(T(\mathcal{V})q).\ p + q \in A\} \\ &= T(\mathcal{U}) + T(\mathcal{V}). \end{aligned}$$

□

As in the proof of the Galvin-Glazer Theorem (3.15) we want to use an idempotent ultrafilter. However in order to take care of the tetris operation, we need to construct a sequence of compatible idempotent ultrafilters as follows:

Lemma 4.5. There idempotent ultrafilters $\mathcal{U}_k \in \gamma\,\mathrm{FIN}_k$, $k \in \mathbb{N}$ such that for all $j > i$:

- $\mathcal{U}_j \leqslant \mathcal{U}_i$[a] and
- $T^{j-i}(\mathcal{U}_j) = \mathcal{U}_i$.

[a] recall that $\mathcal{U} \leqslant \mathcal{V} :\iff \mathcal{U} + \mathcal{V} = \mathcal{V} + \mathcal{U} = \mathcal{U}$

Proof. We construct the sequence recursively. We find $\mathcal{U}_1$ be applying Ellis' Lemma (3.4) to $\gamma\,\mathrm{FIN}_1$. Suppose that $\mathcal{U}_1, \ldots, \mathcal{U}_{k-1}$ have been chosen. We need to find a suitable $\mathcal{U}_k$. Let

$$S_k := \{\mathcal{X} \in \mathrm{FIN}_k \,|T(\mathcal{X}) = \mathcal{U}_{k-1}\}.$$

Claim 1. *$S_k + \mathcal{U}_{k-1}$ is a compact subsemigroup of $\gamma\,\mathrm{FIN}_k$.*

Subproof. For $\mathcal{X}, \mathcal{Y} \in S_k$ we have

$$\begin{aligned} T(\mathcal{X} + \mathcal{U}_{k-1} + \mathcal{Y}) &\overset{\mathcal{X},\mathcal{Y}\in S_k}{=} \mathcal{U}_k + \mathcal{U}_{k-1} + \mathcal{U}_k \\ &\overset{\mathcal{U}_k \leqslant \mathcal{U}_{k-1}}{=} \mathcal{U}_k, \end{aligned}$$

hence $(\mathcal{X} + \mathcal{U}_{k-1}) + (\mathcal{Y} + \mathcal{U}_{k-1}) \in S_k + \mathcal{U}_{k-1}$. S_k is compact, since T is continuous and $S_k + \mathcal{U}_{k-1}$ is compact, since $+$ is right-continuous. ■

By Ellis' Lemma (3.4) we find an idempotent $\mathcal{V} + \mathcal{U}_{k-1} \in S_k + \mathcal{U}_{k-1}$. Set $\mathcal{U}_k := \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1}$. Then

$$\begin{aligned} T(\mathcal{U}_k) &= T(\mathcal{U}_{k-1}) + T(\mathcal{V}) + T(\mathcal{U}_{k-2}) \\ &= \mathcal{U}_{k-2} + \mathcal{U}_{k-1} + \mathcal{U}_{k-2} \\ &\overset{\mathcal{U}_{k-1} \leqslant \mathcal{U}_{k-2}}{=} \mathcal{U}_{k-1}, \end{aligned}$$

$\mathcal{U}_k$ is idempotent since

$$\begin{aligned} \mathcal{U}_k &= \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &\overset{\mathcal{V}+\mathcal{U}_{k-1} \text{ idempotent}}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &\overset{\mathcal{U}_{k-1} \text{ idempotent}}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} \\ &= \mathcal{U}_k + \mathcal{U}_k \end{aligned}$$

and for $l < k$ we have

$$\begin{aligned} \mathcal{U}_k + \mathcal{U}_l &= \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} + \mathcal{U}_l \\ &\overset{\mathcal{U}_{k-1} \leqslant \mathcal{U}_l}{=} \mathcal{U}_{k-1} + \mathcal{V} + \mathcal{U}_{k-1} = \mathcal{U}_k, \end{aligned}$$

hence $\mathcal{U}_k \leqslant \mathcal{U}_l$. □

Proof of Theorem 4.3. Fix $\mathcal{U}_n, n \leqslant k$ as in Lemma 4.5.

Let A_0 be the piece of the partition such that $A_0 \in \mathcal{U}_k$. We define a sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ with $A_i \in \mathcal{U}_k$ and a block sequence $\{x_i\}$ such that

(i) $\forall i.\ x_i \in A_i$ and

(ii) $\forall 1 \leqslant i, j \leqslant n.\ (\mathcal{U}_k x)[T^{k-i}(x_n) + T^{k-j}(x) \in A_n^{\max(i,j)}]$,

where $A_n^l := T^{k-l}[A_n]$.

We do this inductively: A suitable x_0 exists, since for all $i \leqslant j \leqslant k$

$$\begin{aligned}
& (\mathcal{U}_k x).\ x \in A_0 \\
\implies & (T^{k-j}(\mathcal{U}_k)x).\ x \in A_0^j \\
\implies & (\mathcal{U}_j x)x \in A_0^j \\
\implies & ((\mathcal{U}_j + \mathcal{U}_i)x)x \in A_0^j \\
\implies & (\mathcal{U}_j x)(\mathcal{U}_i y)x + y \in A_0^j \\
\implies & (\mathcal{U}_k x)(\mathcal{U}_k y)T^{k-j}(x) + T^{k-i}(y) \in A_0^j
\end{aligned}$$

and similarly for $j \leqslant i$, i.e. $\mathcal{U}$-almost all $x \in A_0$ work.

Suppose that $A_0, \ldots, A_n$ and $x_0, \ldots, x_n$ have been chosen. Let

$$C_n^{i,j} := \{x \in \mathrm{FIN}_k \,|T^{k-i}(x_n) + T^{k-j}(x) \in A_n^{\max(i,j)}\}.$$

We have $C_n^{i,j} \in \mathcal{U}_k$ by (ii), so $A_{n+1} := A_n \cap \bigcap_{i,j} C_n^{i,j} \in \mathcal{U}_k$.

Note that

$$(\mathcal{U}_k x).\ \left((x \in A_{n+1}) \wedge (\mathcal{U}_k y).\ \forall 1 \leqslant i, j \leqslant k.\ T^{k-j}(x) + T^{k-i}(y) \in A_0^{\max(i,j)}\right),$$

so $\mathcal{U}_k$-almost all x can be chosen as x_{n+1}.

Claim 4.3.1. *$\langle\{x_n\}\rangle$ is monochromatic.*

Subproof. We show by induction on p, that

$$T^{k-l_0}(x_{n_0}) + \ldots + T^{k-l_{p-1}}(x_{n_{p-1}}) + y \in A_{n_0}^{\max_i l_i}$$

for all $l_0, \ldots, l_p \leqslant k$, $n_0 < n_2 < n_{p-1}$, $y \in A_{n_p}^{l_p}$.

The case of $p = 0$ follows from the condition (ii).

Suppose we have shown the statement for $p - 1$. Write

$$T^{k-l_0}(x_{n_0}) + \underbrace{T^{k-l_1}(x_{n_1}) + \ldots + T^{k-l_{p-1}}(x_{n_{p-1}}) + y}_{=:y'}.$$

By the induction assumption, we know that $y' \in A_{n_1}^l$, where $l := \max_{i>0} l_i$. Hence there exists $y^* \in A_{n_1}$ such that $y' = T^{k-l}(y^*)$. Since $y^* \in C_{n_0}^{l_0 l}$,

$$T^{k-l_0}(x_{n_0}) + T^{k-l}(y^*) \in A_{n_0}^{\max l_i}.$$

■

□

Corollary 4.6 (Hindman)**.** For every finite coloring of FIN_1, the set of finite subsets of $\mathbb{N}$, there exists an infinite block sequence B, such that $\langle B \rangle$, i.e. the set of unions of a finite, positive number of elements of B, is monochromatic.

Recall that c_0 is the Banach space of real sequences $(x_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} x_n = 0$ and $\|(x_n)_{n\in\mathbb{N}}\| := \sup_{n\in\mathbb{N}} |x_n|$. Let $S^+_{c_0} := \{x \in S_{c_0} | \|x\| = 1 \wedge \forall n.\ x_n \geqslant 0\}$ denote the positive part of the sphere of c_0.

We can view FIN_k as a δ-net in $S^+_{c_0}$ in the following way:

Let δ be such that $\delta = \frac{1}{(1+\delta)^{k-1}}$. Define

$$\begin{aligned} \Phi_k \colon \mathrm{FIN}_{[0,k]} &\longrightarrow S^+_{c_0} \\ f &\longmapsto \begin{pmatrix} \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & \begin{cases} \frac{1}{(1+\delta)^{k-f(n)}} & \text{if } f(n) > 0, \\ 0 & \text{otherwise,} \end{cases} \end{pmatrix} \end{aligned}$$

where $\mathrm{FIN}_{[0,k]} := \bigcup_{i=0}^{k} \mathrm{FIN}_k$. Let $\Delta := \Phi_k[\mathrm{FIN}_k]$. Note that the tetris operation corresponds to scalar multiplication in the sense that if $\lambda \in \mathbb{R}$ and $x \in \mathrm{FIN}_k$ are such that $\lambda\Phi_k(x) \in \Phi_k[\mathrm{FIN}_l]$ for some $l < k$, then $\lambda\Phi_k(x) = \Phi_k(T^{k-l}(x))$.

We obtain:

Corollary 4.7. For every $0 < \delta < 1$, there exists a δ-net Δ in $S^+_{c_0}$ such that for every finite coloring of Δ, there exists an infinite dimensional block subspace X of c_0, such that $X \cap \Delta$ is monochromatic.

5 Infinite Hales-Jewett

5th talk, M. Strohmeier, 2024-05-06

Unfortunately I did not attend. However there exist official notes.

6 Actions on Semigroups, Infinitary Gowers-Hales-Jewett Ramsey Theorem

This section follows [Lup19, sections 1 -4].

6.1 1st Talk

6th talk, Jan Raring, 2024-05-20

I didn't attend. There are official notes.

6.2 1st Talk - Continuation

6th talk, Jan Raring, 2024-05-27

Jan's official notes include a very nice cheat sheet. You'll probably want read those instead of my version, which is incomplete.

Theorem 6.1 (Lupini 2019, [Lup19, p. 3.7])**.** Fix $k \in \omega$. Suppose that α is a k-action on the adequate partial semigroup S given by the map

$$\{0, \dots, k\} \to \mathcal{S}(S), t \mapsto S_t$$

and the subsemigroup $\mathcal{F}_\alpha \subseteq \mathrm{End}(S)$. Suppose that $\xi \in (\gamma S)_\alpha$ is an order-preserving idempotent. Fix a finite coloring c of S and consider its canonical extension to a finite coloring of βS. Fix a sequence $(\psi_n^{(\mathcal{F})})$ of functions

$$\psi_n^{(\mathcal{F})} \colon (S^{\{0,\dots,k\}})^n \to [\mathcal{F}_\alpha]^{<\aleph_0}$$

and a sequence $(\psi_n^{(S)})$ of functions

$$\psi_n^{(S)} \colon (S^{\{0,\dots,k\}})^n \to [S]^{<\aleph_0}$$

such that for every

- $n \geqslant 0$,
- $(y_0, \dots, y_n) \in (S^{\{0,\dots,k\}})^{n+1}$,
- $\tau \in \psi_{n+1}^{(\mathcal{F})}(y_0, \dots, y_n)$,
- $t \in \{0, \dots, k\}$

the following hold:

- $\psi_{n+1}^{(S)}(y_0, \dots, y_n) \supseteq \psi_n^{(S)}(y_0, \dots, y_{n-1})$,
- $\psi_{n+1}^{(S)}(y_0, \dots, y_n) \supseteq \psi_n^{(S)}(y_0, \dots, y_{n-1}) + \tau(y_n(t))$.

Then there exists a sequence (x_n) in $S^{\{0,\dots,k\}}$ such that

- $\forall n \in \omega.\ x_n(t) \in S_t \cap (\varphi_s \circ \psi_n^{(S)})(x_0, \dots, x_{n-1})$
- for every $l < \omega$, $n_0 < \ldots < n_l \leqslant m$ and $t_i \in \{0, \dots, k\}$ for $i \leqslant l$ and $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \dots, x_{n_i - 1})$ for $i \leqslant l$ the color of $\tau_0(x_{n_0}(t_1)) + \ldots + \tau_l(x_{n_l}(t_l))$ is equal to the color of $\xi(\max(\{f_{\tau_i}(t_i) | i \leqslant l\}))$.

Proof. By recursion on $m \in \omega$ we want to define a sequence of functions $x_m \colon \{0, \dots, k\} \to S$ such that

$$x_m(t) \in S_t \cap (\varphi_s \circ \psi_m^{(S)})(x_0, \dots, x_{m-1})$$

and such that for every $m \in \omega$, the following two conditions hold for every

- $l \leqslant m$,
- $n_0 < n_1 < \ldots < n_l \leqslant m$,
- $t_i \in \{0, \ldots, k\}$ for $i \leqslant l+1$,
- $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \ldots, x_{n_i-1})$ for $i \leqslant l$.

(abbreviated $Q(m)$)

(I_m) the color of
$$\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(x_{n_l}(t_l))$$
is equal to the color of $\xi(\max(\{f_{\tau_i}(t_i) | i \leqslant l\}))$,

(II_m) the color of
$$\tau_0(x_{n_0}(l)) + \ldots + \tau_l(x_{n_l}(t_l)) + \xi(t_{l+1})$$
is equal to the color of
$$\xi(\max(\{f_{\tau_i}(t_i) | i \leqslant l\} \cup \{t_{l+1})).$$

$m = 0$: In this case $(S^{\{0,\ldots,k\}})^0$ is a singleton, the $\psi_0^{(S)}$ select a finite subset A_0 of S and the $\psi_0^{(\mathcal{F})}$ select a finite subset $\mathcal{F}_0$ of $\mathcal{F}_\alpha$.

We need to find a function
$$x_0 \colon \{0, \ldots, k\} \to S$$
such that $x_0(t) \in S_t \cap \varphi_s(A_0)$ and

- $c(\tau_0(x_0(t))) = c(\xi(f_{\tau_0}(t)))$
- for all $t_0, t_1 \in \{0, \ldots, k\}$ and $\tau_0 \in \mathcal{F}_0$ $c(\tau_0(x_0(t_0)) + \xi(t_1)) = c(\xi(\max(\{f_{\tau_0}(t_0), t_1\}))$.

Fixing $t \in \{0, \ldots, k\}$ we claim that
$$\begin{aligned}
&(\xi(t)s)[s \in S_t \cap \varphi_s(A_0) \wedge \\
&\quad c(\tau_0(s)) = c(\xi(f_{\tau_0}(t))) \wedge \\
&\quad \forall \tau_0 \in \mathcal{F}_0 \forall t_1 \in \{0, \ldots, k\}. \\
&\quad\ (c(\tau_0(s)) + \xi(t_1)) = c(\xi(\max\{f_{\tau_0}(t), t_1\}))) \\
&\quad]
\end{aligned}$$

(omitted)

$m \rightsquigarrow m+1$. Suppose that a sequence as above has been the defined up to m such that (I_m) and (II_m) hold. From (II_m) and the fact that $\xi(t)$ is an order-preserving idempotent, we get that (III_m): for every $l \leqslant m$, $n_0 < \ldots < n_l \leqslant m$, $t_i \in \{0, \ldots, k\}$ for $i \leqslant l+2$, $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \ldots, x_{n_i-1})$ for $i \leqslant l$ $t = \max\{f_{\tau_i}(t_i) | i \leqslant l\} \cup \{t_{i+1}, t_{i+2}\}$ we have $c(\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(x_{n_l}(t_l)) + \xi(t_{l+1}) + \xi(t_{l+2}) = c(\xi(t))$.

Fix $t \in \{0, \ldots, k\}$. Using (II_m) and (III_m) we want to show that

$$
\begin{aligned}
&(\xi(t)s)[\forall l \leqslant m, \\
&\quad n_0 < \ldots < n_l \leqslant m, \\
&\quad \tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \ldots, x_{n_i - 1}), i \leqslant l, \\
&\quad t_i \in \{0, \ldots, k\} \\
&\quad \tau \in \psi_{n+1}^{(\mathcal{F})}(x_0, \ldots, x_m) : \\
&\quad c(\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(\tau_{n_l}(t_l)) + \tau(s)) = c(\xi\,(\max\,(\{f_{\tau_i}(t_i) | i \leqslant l\} \cup \{f_\tau(t)\}))) \quad (1) \\
&\quad \wedge c(\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(\tau_{n_l}(t_l)) + \tau(s) + \xi(t_{l+2}) = c(\xi\,(\max\,(\{f_{\tau_i}(t_i) | i \leqslant l\} \cup \{f_\tau(t), t_{l+2}\}))) \quad (2)
\end{aligned}
$$

Subproof. From (II_m) we get that for all

- $l \leqslant m$,
- $n_0 < \ldots < n_l \leqslant m$,
- $t_i \in \{0, \ldots, k\}$ for $i \leqslant l + 1$,
- $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \ldots, x_{n_i})$, $i \leqslant l$

we have

$$c(\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(x_{n_l}(t_l)) + \xi(t_{l+1})) = c(\xi(\max(\{f_{\tau_i}(t_i) | i \leqslant l\} \cup \{t_{l+1}\}))).$$

Now choose $\tau \in \psi_{n+1}^{(\mathcal{F})}(x_0, \ldots, x_m)$, set $t_{l+1} := f_\tau(t)$. Then

$$c(\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(x_{n_l}(t_l)) + \xi(f_\tau(t))) = c(\max\,(\{f_{\tau_i}(t_i) | i \leqslant l\} \cup \{f_\tau(t)\})).$$

In particular

$$(\xi(f_\tau(t))s)[c(\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(x_{n_l}(t_l)) + s) = c(\xi(\max\{f_{\tau_i}(t_i) | i \leqslant l\} \cup \{f_\tau(t)\}))],$$

i.e.

$$(\tau(\xi(t))s)[\ldots]$$

hence

$$(\xi(t)s)[c(\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(x_{n_l}(t_l)) + \tau(s)) \ldots].$$

■

Now we apply (III_m) to

$$c(\tau_0(x_{n_0}(t_0))) + \ldots + \tau_l(x_{n_l}(t_l)) + \xi(f_{\tau(t))})) = c(\xi(\max(\{f_{\tau_i(t_i)|i \leqslant l} \cup \{f_\tau(t)\}\})).$$

We obtain

$$
\begin{aligned}
& c(\tau_0(x_{n_0}(t_0)) + \ldots + \tau_l(x_{n_l}(t_l)) + \xi(f_\tau(t)) + \xi(t_{l+2})) \\
= \; & c(\xi\,(\max\{f_{\tau_i}(t_i) | i \leqslant l\} \cup \{f_\tau(t), t_{l+2}\})) \\
\iff \; & (\xi(t)s)[\ldots].
\end{aligned}
$$

Thus we can choose $x_{m+1}(t)$ satisfying (I_{m+1}) and (II_{m+1}). □

6.3 2nd Talk

7th talk, Shujie Yang, 2024-05-27

My notes on this talk are very much unfinished. There don't seem to be official notes.

Definition 6.2. Let X be a semigroup and α a k-action on X.

We say that α is a **Ramsey k-action** iff each f_τ is a regressive function and $(X)_\alpha$ is not empty.

Definition 6.3. Suppose that X is an *adequate* partial semigroup and α a k-action on X. We says that α is a **Ramsey k-action** iff f_τ is regressive and for $S_0 \overset{\text{finite}}{\subseteq} S$, $\mathcal{F}_0 \overset{\text{finite}}{\subseteq} \mathcal{F}_\alpha$ and c a finite coloring of S, there exists $x\colon \{0, \dots, k\} \to S$ such that

1. $\forall t.\ x(t) \in S_t \cap \varphi_S(S_0)$
2. for any $\tau \in \mathcal{F}_0$, the color of $\tau(x(t))$ depends only on $f_\tau(t)$.

Example 6.4. Let $S = \mathrm{FIN}_{\leqslant k}$.

We have $\mathrm{FIN}_k \leqslant \mathrm{FIN}_{k-1} \leqslant \dots \leqslant \mathrm{FIN}_0$. Consider generalized tetris functions.

Let f be a regressive function, $b \in \mathrm{FIN}_{\leqslant k}$. Define a k-action by $\tau_f(b)(n) = f(b(n))$.

Let $x(t)$ be the function with support S_0 and value t.

Proposition 6.5. Let X be a semigroup and α a k-action on X. If α is a Ramsey k-action on X, then there exists $\xi \in (X)_\alpha$ such that ξ is idempotent and order preserving.

Proof. omitted. □

Theorem 6.6 ([Lup19, p. 3.9])**.** Let S be an adequate partial semigroup and α a k-action such that each f_τ is regressive.

Then the following are equivalent:

- α is Ramsey.
- The induced k-action α on γS is Ramsey.
- long statement similar to Theorem 6.1.

Proof. (3) $\implies$ (1): let $n = 1$. (2) $\implies$ (3): (2) is the condition needed in the proof of Theorem 6.1.

(2) $\implies$ (1): Fix some $\xi \in (\gamma S)_\alpha$. Consider $S_0 \overset{\text{finite}}{\subseteq} S$, $\mathcal{F}_0 \overset{\text{finite}}{\subseteq} \mathcal{F}_\alpha$ and a coloring c.

We have a canonical extension of c to γS. Then $(\xi(t)s)\,(s \in S_t \cap \varphi_S(S_0) \wedge c(s) = c(\xi(f_\tau(t))))$.

(1) $\implies$ (2):

Given $S_0 \overset{\text{finite}}{\subseteq} S$, $\mathcal{F}_0 \overset{\text{finite}}{\subseteq} \mathcal{F}_\alpha$ and a coloring c, we need to find a suitable x.

We find $\xi\colon \{0, \ldots, k\} \to \gamma S$ such that for any $\tau \in \mathcal{F}_\alpha$ and any coloring c on S, $\tau \circ \xi(t)$ only depends on $f_\tau(t)$. Using this we can deduce $\tau \circ \xi(t) = \xi \circ f_\tau(t)$.

Fix $\mathcal{F}_0 \subseteq \mathcal{F}_\alpha$ and c. Let $x_i(t) \subseteq S_t$ be the x corresponding to $\mathcal{F}_0, c$ and S_i. Finish proof

$\square$

6.4 3rd Talk

8th talk, Kwiatkowska, 2024-06-03

Definition 6.7. The **tensor product** of ultrafilters $\mathcal{U} \in \mathcal{P}(A)$, $\mathcal{V} \in \mathcal{P}(B)$, $\mathcal{U} \otimes \mathcal{V}$, on $A \times B$ is defined by letting $C \in \mathcal{U} \otimes \mathcal{V} :\iff (\mathcal{U}a)(\mathcal{V}b)(a, b) \in C$.

Theorem 6.8 (Bergelson-Hindman-Williens)**.** Let $m \in \mathbb{N}$. Suppose that $(S_i, +_i), i < m$ are semigroups and $p_i \in \gamma S_i$ idempotent.

Consider

$$A \in p_0 \otimes \ldots \otimes p_{m-1}$$

Then for each $i < m$ there is a sequence $(y_i, n)_{n \in \mathbb{N}}$ in S_i such that

- $\{(\sum_{d \in F_s}{}^{(S_i)} y_{s,d})_{s<m} : F_0 < F_1 < \ldots < F_{m-1} \overset{\text{finite}}{\subseteq} \mathbb{N}\} \subseteq A$. ($A < B$ denotes $\max(A) < \min(B)$).

Note that for $m = 1$, $(S, +) = (\mathbb{N}, +)$ we recover Hindman's theorem.

Why do we need $F_0 < F_1 < \ldots < F_{m-1}$? Consider $m = 2$, and $S_0 = S_1 = (\mathbb{N}, +)$. Consider the coloring

$$\begin{aligned} c\colon (\mathbb{N}, +) \times (\mathbb{N}, +) &\longrightarrow \{\pm 1\} \\ (i, j) &\longmapsto [i > j] - [i \leqslant j]. \end{aligned}$$

For each $i < m$, let $(S_i, +_i)$ be an adequate partial semigroup. Let $k \in \mathbb{N}$. For every $i < m$, we define a k-action α_i on $(S_i, +_i)$. As before α_i extends to γS_i.

Theorem 6.9. Suppose $\xi_i \in (\gamma S_i)_{\alpha_i}$ is an order-preserving idempotent. Let $S := S_0 \times \ldots \times S_{m-1}$ and c a coloring of S.

Suppose that there are sequences

- $(\psi^{(S)}_{i,n})$, $\psi^{(S)}_{i,n} : (\{S_i^{\{0,\dots,k\}})^{n+1} \to [S_i]^{<\aleph_0}$,
- $(\psi^{(\mathcal{F})}_{i,n})$, $\psi^{\mathcal{F}}_{i,n} : (S_i^{\{0,\dots,k\}})^{n+1} \to [\mathcal{F}_{\alpha_i}]^{<\aleph_0}$,

such that for all n, i and $(y_0, \dots, y_n) \in (S_i^{\{0,\dots,k\}})$, $\psi^{(S)}_{i,n+1}(y_0, \dots, y_n)$ contains $\psi^{(S)}_{i,n}(y_0, \dots, y_{n-1})$ and contains $\psi^{(S)}_{i,n}(y_0, \dots, y_{n-1}) + \tau(y_n(t))$.

Then for all $i < m$ there is a sequence $(x_{i,n})_{n\in\mathbb{N}}$ of functions $x_{i,n} \colon \{0, \dots, k\} \to s_i$ such that

- $x_{i,n}(t) \in S_{i,t} \cap \varphi_{S_i}\left(\psi^{(S)}_{i,n}(x_{i,0}, \dots, x_{i,n-1})\right)$,
- for any $i < m$, and
  - finite subsets $F_0 < \dots < F_{m-1} \overset{\text{finite}}{\subseteq} \mathbb{N}$,
  - $(t_{i,n})_{n\in\mathbb{N}}$ in $\{0, \dots, k\}$,
  - $(\tau_{i,n})_{n\in\mathbb{N}}$ in $\mathcal{F}_{\alpha_i}$, such that $\forall n.\ \tau_{i,n} \in \psi^{(\mathcal{F})}_{i,n}(x_{i,0}, \dots, x_{i,n-1}$,
  - $t_s := \max\{f_{\tau_{s,d}}(t_{s,d}) : d \in F_s\}, s \in m$,

  we have that the color of

  $$\left(\sum_{d\in F_s}^{S_s} \tau_{s,d}(x_{s,d}(t_{s,d}))\right)_{s<m}$$

  is the same as the color of

  $$\xi_0(t_0) \otimes \xi_1(t_1) \otimes \dots \otimes \xi_{m-1}(t-1).$$

The proof is the same as last time (with even more indices).

Theorem 6.10 (Lupini)**.** Let $k \in \mathbb{N}$, c a coloring of $\mathrm{FIN}_{\leqslant k}$, (d_n) in $\mathbb{N}$. Then there exist (b_n) in FIN_k such that

- $\mathrm{dom}(b_n) > d_n$,
- $\mathrm{dom}(b_{n+1}) > \mathrm{dom}(b_n)$ and
- $\forall l \in \mathbb{N}, n_0 < n_1 < \dots < n_l$, $f_i \colon \{0, \dots, k\} \to \{0, \dots, k\}$ regressive $i \leqslant l$ the color of $\tau_{f_0}(b_{n_0}) + \dots + \tau_{f_{n_i}}(b_{n_i})$ depends only on $\max\{f_i(k) : i \leqslant l\}$.

Theorem 6.11. Let $m \in \mathbb{N}$, $k \in \mathbb{N}$. Fix a coloring c of $(\mathrm{FIN}_{\leqslant k})^m$. For any (d_n) there are sequences $(b_{i,n})_{n\in\mathbb{N}}$, $i < m$, in $\mathrm{FIN}_{\leqslant k}$ such that

- $\mathrm{dom}(b_{i,n}) > d_n$, $\mathrm{dom}(b_{i,n+1}) > \mathrm{dom}(b_{i,n})$ and

- for any finite $F_0 < F_1 < \ldots < F_{m-1} \overset{\text{finite}}{\subseteq} \mathbb{N}$ and regressive homomorphisms $f_{i,d} \colon \{0, \ldots, k\} \to \{0, \ldots, k\}$ such that $t_s = \max\{f_{s,d}(k), d \in F_s\}$, $s < m$ the color of

$$\left(\sum_{d \in F_s} \tau_{f_{s,d}}(b_{s,d}) \right)_{s<m}$$

depends only on $(t_s)_{s<m}$

6.5 4th Talk

9th talk, Shujie Yang, 2024-06-10

I didn't take notes and official notes don't seem to exist.

7 Galvin-Prikry and the Ellentuck Topology

7.1 1st Talk

10th talk, Prof. Kwiatkowska, 2024-07-01

Notation 7.0.4. $[\mathbb{N}]^{\aleph_0} := \{A \subseteq \mathbb{N} : |A| = \aleph_0\} \subseteq 2^{\mathbb{N}}$, where we identify $\mathcal{P}(\mathbb{N})$ and $2^{\mathbb{N}}$ in the natural way.

From the topology of $2^{\mathbb{N}}$ we get an induced topology on $[\mathbb{N}]^{\aleph_0}$.

Theorem 7.1 (Galvin-Prikry)**.** Let $[\mathbb{N}]^{\aleph_0} = P_0 \cup \ldots \cup P_{k-1}$ a coloring, where all the P_i are Borel (in the topology of $2^{\mathbb{N}}$).

Then we can find an infinite set $H \in [\mathbb{N}]^{\aleph_0}$, such that $\exists i.\ [H]^{\aleph_0} \subseteq P_i$.

Example 7.2. The assumption that the B_i are "definable" (not necessarily Borel, but in some sense well behaved) is required:

Let $k = 2$. Enumerate all $[\mathbb{N}]^{\aleph_0}$ into $(H_\xi)_{\xi < 2^{\aleph_0}}$.

By transfinite recursion, find disjoint $A_\xi, B_\xi \in [\mathbb{N}]^{\aleph_0}$ for all ξ such that $A_\xi \cup B_\xi \subseteq H_\xi$. Let $P_0 := \{A_\xi : \xi \in 2^{\aleph_0}\}$ and $P_1 := [\mathbb{N}]^{\aleph_0} \backslash P_0 \supseteq \{B_\xi : \xi \in 2^{\aleph_0}\}$.

We'll usually use lower case letters for finite sets and upper case for infinite sets.

Definition 7.3. Let $a \subseteq \mathbb{N}$ finite and $A \subseteq \mathbb{N}$ infinite with $a < A$ (i.e. $\max(a) < \min(A)$). Set $[a, A] := \{X \in [\mathbb{N}]^{\aleph_0} : a \subseteq X \subseteq a \cup A\}$.

The topology generated by sets of the form $[a, A]$ is called the **Ellentuck topology**.

Note that $[\varnothing, A] = [A]^{\aleph_0}$.

Remark 7.3.5. $[a, A] \subseteq [b, B]$ iff $a \supseteq b$, $a \backslash b \subseteq B$ and $A \subseteq B$.

Notation 7.3.6. Let $X \backslash\!\backslash n := \{x \in X : x > n\}$. Note that this is not the same as $\backslash$ for ordinals, e.g. $\mathbb{N}\backslash 4 = \mathbb{N}\backslash\!\backslash 3$!

Claim 1. *The sets of the form $[a, A]$ form the basis of a topology on $[\mathbb{N}]^{\aleph_0}$.*

Proof. Indeed if $X \in [a, A] \cap [b, B]$, then

$$X \in [a \cup b, X \backslash\!\backslash \max(a \cup b)] \subseteq [a, A] \cap [b, B].$$

□

Claim 2. *The Ellentuck topology refines the usual topology on $[\mathbb{N}]^{\aleph_0} \subseteq 2^{\aleph_0}$.*

Proof. This is trivial. □

Definition 7.4. A set $X \in [\mathbb{N}]^{\aleph_0}$ is **completely Ramsey** if for any $a < A$, there is $B \subseteq A$ such that either $[a, B] \subseteq X$ or $[a, B] \in X^c$.

Theorem 7.5 (Ellentuck 1974)**.** Let $X \in [\mathbb{N}]^{\aleph_0}$. Then X is completely Ramsey iff it has the BP (wrt. the Ellentuck topology).

As a corollary we get:

Proof of Theorem 7.1. It suffices to consider the case of $k = 2$. Let $[\mathbb{N}]^{\aleph_0} = P_0 \cup P_1$, where P_0, P_1 are completely Ramsey.

Consider $A = [\varnothing, \mathbb{N}]$. There is $B \in [\mathbb{N}]^{\aleph_0}$ such that $[B]^{\aleph_0} \subseteq P_0$ or $[B]^{\aleph_0} \subseteq P_0^c = P_1$. □

Proof of Theorem 7.5.

Claim 7.5.1. *Every Ellentuck-open set is completely Ramsey.*

Proof of Claim 7.5.1. Let U be open.

$[a, A]$ is **good** iff for some $B \subseteq A$ it holds that $[a, B] \subseteq U$. Otherwise it is **bad**. $[a, A]$ is **very bad** if it is bad and $\forall n \in A$. $[a \cup \{n\}, A \backslash\!\backslash n]$ is bad.

Note that if $[a, A]$ is (very) bad and $B \subseteq A$, then $[a, B]$ is also (very) bad.

Claim 7.5.1.1. *If $[a, A]$ is bad, then there exists $B \subseteq A$ such that $[a, B]$ is very bad.*

Subproof. Suppose that $[a, A]$ is a counter example. So there is $n_0 \in A$, such that $[a \cup n_0, A \setminus n_0]$ is good. So there is $B_0 \subseteq A \setminus n_0$ such that $[a \cup \{n_0\}, B_0] \subseteq U$.

Since $[a, B_0]$ is not very bad, we can continue with this, i.e. find $n_1 \in B_0$, $B_1 \subseteq B_0 \setminus n_1$ such that $[a \cup \{n_1\}, B_1] \subseteq U$.

We get $n_0 < n_1 < \ldots$ and $A \supseteq B_0 \supseteq B_1 \supseteq \ldots$. Set $B := \{n_0, n_1, \ldots\}$. Then $[a, B] \subseteq U$. But $[a, A]$ is bad $\lightning$. ■

We want to show that U is completely Ramsey. So given some $[a, A]$, we need to find $B \subseteq A$, such that $[a, B] \subseteq U$ or $[a, B] \subseteq U^c$. If $[a, A]$ is good, this is trivial. If $[a, A]$ is bad, we find $B \subseteq A$ such that $[a, B] \subseteq U^c$ as follows:

Repeatedly using the claim we find $A \subsetneq B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \ldots$, $n_i := \min(B_i)$ such that

$$\forall b \subseteq \{n_0, \ldots, n_{i-1}\}.\ [a \cup b, B_i] \text{ is very bad.}$$

By the claim we find B_0, such that $[a, B_0]$ is very bad. Then $[a \cup \{n_0\}, B_0 \setminus \{n_0\}]$ is bad. Take $B_1 \subseteq B_0 \setminus n_0$ such that $[a \cup \{n_0\}, B_1]$ is very bad. Then $[a, B_1]$ is very bad. ...

Let $B := \{n_0, n_1, \ldots\}$. Then $[a, B] \subseteq U^c$: Otherwise since U is open, there is $[a', B'] \subseteq [a, B]$ such that $[a', B'] \subseteq U$. So $a' = a \cup b$, where $b = \{n_0, \ldots, n_i\}$ for some i and $B' \setminus n_i \subseteq B \setminus n_i$. So $[a \cup b, B' \setminus n_i] \subseteq U$. Therefore $[a \cup b, B_i \setminus n_i]$ is good $\lightning$. □

Next steps:

- nwd sets (not hard)
- meager sets (not hard)
- BP sets.

□

7.2 2nd Talk

11th talk, Prof. Kwiatkowska, 2024-07-08

Recall that $X \subseteq [\mathbb{N}]^{\aleph_0}$ is **completely Ramsey** iff for every $< A$, $A \overset{\text{infinite}}{\subseteq} \mathbb{N}$ there is $B \subseteq A$ with $[a, B] \subseteq X$ or $[a, B] \subseteq X^c$, where

$$[a, A] = \{X \in [\mathbb{N}]^{\aleph_0} : a \subseteq X \subseteq a \cup A\}.$$

Continuation of proof of Theorem 7.5. We have seen last time that open sets are completely Ramsey. See [Kec95, chapter 19] for a complete proof. (the other parts are not as much work).

" $\Longrightarrow$ " If X is completely Ramsey, then it has the Baire property.

Claim 7.5.3. *$Y = X \setminus \operatorname{Int}(X)$ is nwd.*

Subproof. Otherwise there is some basic open $[a, A] \subseteq \overline{Y}$. Since X is completely Ramsey, there exists and infinite subset $B \subseteq A$ such that $[a, B] \subseteq X$ or $[a, B] \subseteq X^c$.

1. $[a, B] \subseteq X^c$: This can not happen as $[a, B]$ is open, hence $[a, B] \cap Y \neq \varnothing$, since $[a, B] \subseteq \overline{Y}$.
2. $[a, B] \subseteq X$:

   Again, since $[a, B]$ is open, we get $[a, B] \subseteq \operatorname{Int}(X)$. So $[a, B] \cap Y = \varnothing$. But this contradicts $[a, B] \subseteq \overline{Y}$ $\lightning$

■

□

7.2.1 Application to Banach spaces

Definition 7.6. Let (x_n) be a sequence in a real Banach space X. We say that (x_n) is equivalent to the unit basis[a] of $\ell^1 = \{f \colon \mathbb{N} \to \mathbb{R} | \sum_{i=0}^{\infty} |f(i)| < \infty\}$ iff there are $a, b > 0$ such that $\forall n \in \mathbb{N}.\ \exists c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$

$$a \sum_{i=0}^{n-1} |c_i| \leqslant \left\| \sum_{i=0}^{n-1} cix_i \right\| \leqslant b \sum_{i=0}^{n-1} |c_i|.$$

[a] $(0, 0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots)$, $n \in \mathbb{N}$

Note that this means that $\ell^1 \hookrightarrow X$.

Definition 7.7. Let $S \neq \varnothing$. Let $\ell^\infty := \{f \colon S \to \mathbb{R} | \exists b \in \mathbb{R}.\ \forall s \in S.\ |f(s)| < b\}$, with $\|f\|_\infty := \sup\{|f(x)| : x \in S\}$.

Theorem 7.8 (Rosenthal)**.** Let (f_n) be a bounded sequence in $\ell^\infty(S)$. Then there is a subsequence $(f_{n_k})_k$ such that either

- (f_{n_k}) is pointwise convergent.
- (f_{n_k}) is equivalent to the unit basis of ℓ^1.

Corollary 7.9. Suppose that X is a real Banach space. Then the following are equivalent:

- ℓ^1 does not embed in X.
- Every bounded sequence (x_n) in X has a weakly Cauchy subsequence,

i.e. a subsequence (x_{n_k}) such that for any $x^* \in X^*$, $(x * (x_{n_k})_k$ converges.

Definition 7.10. Let $A, B \subseteq S$.

- (A, B) is **disjoint** iff $A \cap B = \varnothing$.
- $(A_n, B_n)_n$ is **independent** iff (A_n, B_n) is disjoint for all n and for all $F, G \overset{\text{finite}}{\subseteq} \mathbb{N}$, with $F \cap G = \varnothing$ we have
$$\bigcap_{n \in F} A_n \cap \bigcap_{n \in G} B_n \neq \varnothing.$$
- $(A_n, B_n)_n$ is **convergent** iff for all $x \in S$ either
  - for all but finitely many n, $x \notin A_n$ or
  - for all but finitely many n, $x \notin B_n$.

Lemma 7.11. If $(f_n) \in \ell^\infty(S)$ is uniformly bounded, $r < s$, $r, s \in \mathbb{Q}$,
$$A_n^{r,s} := \{x \in S : f_n(x) < r\},$$
$$B_n^{r,s} := \{x \in S : f_n(x) > s\},$$
and (A_n, B_n) is independent, then (f_n) is equivalent to the unit basis of ℓ^1.

Proof. Fix n and $c_0, \ldots, c_{n-1}$. We need to find a, b such that
$$\sum_{i=0}^{n-1} a|c_i| \leqslant \left\| \sum_{i=0}^{n-1} c_i f_i \right\|_\infty \leqslant \sum_{i=0}^{n_1} |c_i| b.$$
Let b be such that $\forall n.\ \|f_n\|_\infty \leqslant b$.

We show that $a := \frac{s-r}{2}$ works. Let
$$F := \{i < n : c_i \geqslant 0\},$$
$$G := \{i < n : c_i < 0\}.$$

By independence there exist
$$x \in \bigcap_{i \in F} A_i \cap \bigcap_{i \in G} B_i,$$
$$y \in \bigcap_{i \in f} A_i \cap \bigcap_{i \in F} B_i.$$

Let

$$c := \sum c_i f_i(y) \geqslant \sum_{i \in F} |c_i| s - \sum_{i \in G} |c_i| r.$$

$$d := \sum c_i f_i(x) \leqslant \sum_{i \in F} |c_i| r - \sum_{i \in G} |c_i| s.$$

So $c - d \geqslant (s - r) \sum_{i=0}^{n-1} |c_i|$.

We want to show

$$\left(\frac{s-r}{2}\right) \sum_{i=0}^{n-1} |c_i| \leqslant \| \sum c_i f_i \|_\infty.$$

It is

$$|c| = \left| \sum_{i<n} c_i f_i(y) \right|,$$

$$|d| = \left| \sum_{i<n} c_i f_i(x) \right|.$$

So

$$\frac{c-d}{2} = \frac{\sum_{i<n} c_i f_i(y) - \sum_{i<n} c_i f_i(x)}{2} \leqslant \sup_{s \in S} | \sum_{i<n} c_i f_i(s)|.$$

□

Lemma 7.12. If $(A_n^{r,s}, B_n^{r,s})_n$ is convergent for all $r < s$, then (f_n) is convergent pointwise.

Proof. Suppose that (f_n) does not converge pointwise. Then there is $x \in S$ such that $\liminf f_n(x) < r < s < \limsup f_n(x)$ for some $r < s$.

Considering

$$A_n^{r,s} = \{x \in S : f_n(x) < r\},$$
$$B_n^{r,s} = \{x \in S : f_n(x) > s\},$$

we get that $(A_n^{r,s}, B_n^{r,s})_n$ is not convergent. □

Lemma 7.13. Every sequence (A_n, B_n) contains a convergent subsequence or an independent sequence.

Proof. Let $P \subseteq [\mathbb{N}]^{\aleph_0}$ by defined by

$$\{n_0 < n_1 < \ldots\} \in P \iff \forall_k \left[\bigcap_{\substack{i<k \\ i \text{ even}}} A_{n_i} \cap \bigcap_{\substack{i<k \\ i\text{odd}}} B_{n_i} \neq \varnothing \right].$$

P is closed. Hence by Ellentuck, there exists $H \in [\mathbb{N}]^{\aleph_0}$ such that either $[H]^{\aleph_0} \subseteq P$ or $[H]^{\aleph_0} \subseteq P^c$.

- $[H]^{\aleph_0} \subseteq P$:

  Let $H = \{m_0 < m_1 < \ldots\}$. Then $((A_{m_{2i+1}}, B_{m_{2i+1}}))$ is independent.

  Just write it down, it is a bit technical, but not hard

- $[H]^{\aleph_0} \subseteq P^c$:

  Suppose that (A_n, B_n) does not converge, i.e. there exists some $x \in S$ such that $x \in A_n$ for infinitely many n and $x \in B_m$ for infinitely many m.

  Let $I := \{m_i : x \in A_i\}$, $J := \{m_j : x \in B_j\}$. I and J are infinite and disjoint, so we find $n_0 < n_1 < \ldots$ in $[H]^{\aleph_0}$, such that $\{n_0, n_2, n_4\} \subseteq I$, $\{n_1, n_3, n_5\} \subseteq J$.

  But $n_0 < n_1 < \ldots \in P$ $\lightning$

$\square$

Proof of Theorem 7.8. The theorem follows directly from Lemma 7.11, Lemma 7.12 and Lemma 7.13. $\square$

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