# Seminar: Model Theory and Combinatorics

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# Contents

These are my notes on the Seminar Model Theory and Combinatorics taught by Prof. Martin Hils and Dr. Rob Sullivan in the summer term 2024 at the University Münster.

**Warning 0.1.** This is not an official script.

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# 1 Introduction and Ultrahomogeneity

1st talk, Rob Sullivan, 2024-04-09

**Theorem 1.1** (Erdős-Rényi)**.** Slogan: "Any two countably infinite random graphs are almost surely isomorphic".

Let $0 < p < 1$. Then there exists a countably infinite graph $R$ such that if we produce a graph $\Gamma$ with $V(\Gamma) := \mathbb{N}$ and for each pair of vertices we put an edge with probability $p$ then $\mathbb{P}[\Gamma \cong R] = 1$.

This graph $R$ is called **the random graph**.

*Proof of Theorem 1.1.* We say that a graph $\Gamma$ has the **witness property** (WP) iff for all pairs $(U, V)$ of finite disjoint subsets of $V(\Gamma)$, we have $w \in \Gamma \setminus (U \cap V)$, such that $w$ is adjacent to all elements of $U$ and none of $V$.

Any graph with the witness property is infinite.

The proof is split into two steps:

**Claim 1.1.1.** $\mathbb{P}[\Gamma \textit{ has } WP] = 1$.

**Claim 1.1.2.** *Any two countably infinite graphs with the witness property are isomorphic.*

*Proof of Claim 1.1.1.* We have

$$\begin{aligned}\mathbb{P}[\Gamma \text{ does not have WP}] &= \mathbb{P}[\exists U, V \text{ with no witness}] \\ &\leqslant \sum_{(U,V)} \mathbb{P}[(U,V) \text{ has no witness}]\end{aligned}$$

For a fixed node $w \in V(\Gamma) \setminus (U \cap V)$, the probability that $w$ is not a witness for $(U, V)$ is $1 - p^{|U|}(1-p)^{|V|}$, hence $\mathbb{P}[(U,V) \text{ have no witness}] = 0$. □

**Explicit construction** We give an explicit construction of a graph with the WP:[1] Let $\mathbb{N}$ be the set of nodes. For $x, y \in \mathbb{N}$, we set $x \sim y$ iff the $x^{\text{th}}$ digit of the binary expansion of $y$ is 1 or the $y^{\text{th}}$ digit of the binary expansion of $x$ is 1.[2]

For $U, V$ finite and disjoint, let $n := \max(U \cap V)$ and take $w \geqslant 2^n$ such that the binary digits of $w$ are set appropriately.

*Proof of Claim 1.1.2.* Now let $\Gamma$ and $\Gamma'$ be countable graphs with the WP. We prove by "back-and-forth"[3], that they are isomorphic:

Enumerate $V(\Gamma)$ as $m_0, m_1, \ldots$ and $V(\Gamma')$ as $m'_0, m'_1, \ldots$ We construct a sequence of partial isomorphisms $f_n$ (i.e. isomorphisms of a finite subset of $\Gamma$ to a finite subset of $\Gamma'$) such that

(i) $f_{n+1} \supseteq f_n$,

(ii) $m_t \in \operatorname{dom} f_{2t}$,

(iii) $m'_t \in \operatorname{dom} f_{2t+1}$.

Given such partial isomorphisms, we can take $f := \bigcup f_t$.

Suppose we already have $f_{2t-1}$. If $m_t \in f_{2t-1}$ let $f_{2t} := f_{2t-1}$. Otherwise let $U$ be the neighbourhood of $m_t$ and $V := (\operatorname{dom} f_{2t-1}) \backslash U$. Choose a witness $w$ for $f_{2t-1}(U), f_{2t-1}(V)$ and set $f_{2t}(m_t) := w$. □

□

The proof of Claim 1.1.2 was very similar to the proof of

**Theorem 1.2** (Cantor)**.** Any two countably infinite DLOWEs[a] are isomorphic.

[a] dense linear orders without endpoints

Dense and without endpoints is equivalent to:

For any $A \subseteq_{\text{fin}} M$ and any partition $A = U \cup V$, where $U < V$[4] there exists $w \in M$ with $U < w < V$.

**Idea.** *We want to generalize this.*

Let $\mathcal{L}$ be a countable relational language and $M$ and $\mathcal{L}$-structure.

**Definition 1.3.** The age of $M$ is the class of finite structures embeddable into $M$.

[1] This is not necessary for the proof.
[2] $0 \in \mathbb{N}$ and indices start at 0
[3] maybe it should be called "forth-and-back"
[4] i.e. $\forall u \in U, v \in V.\ u < v$

We say that $M$ has the **extension property** (**EP**) iff

$$\begin{aligned}&\forall A \subseteq_{\text{fin}} M.\\ &\quad \forall f\colon A \to B \text{ embedding where } B \in \operatorname{Age}(M).\\ &\quad\quad \exists h\colon B \to M \text{ embedding.}\\ &\quad\quad\quad \forall a \in A.\ hf(a) = a\end{aligned}$$

$$\begin{tikzcd} A \arrow[r, phantom, "\subseteq"] \arrow[d, "f"'] & M \\ B \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$

**Example 1.4.** A simple induction shows that the age of $R$ is the set of all finite graphs. The random graph has the extension property.

The extension property was the key ingredient to the proof of Theorem 1.1.

**Proposition 1.5.** Let $M, M'$ be countable $\mathcal{L}$-structures with the same age and the EP. Then $M \cong M'$.

*Proof.* This is same as the proof of Claim 1.1.2. ☐

**Remark 1.5.1.** One needs to be more careful if $\mathcal{L}$ contains constants and function symbols.

**Proposition 1.6.** Let $M$ and $M'$ be countable $\mathcal{L}$-structures with the same age and the EP.

Then any partial isomorphism $A \to A'$ where $A$ and $A'$ are finite substructures of $M$ resp. $M'$ extends to an isomorphism $M \to M'$.

*Proof.* Use $A \to A'$ as a starting point for the back-and-forth in the proof of Proposition 1.5. ☐

**Definition 1.7.** An $\mathcal{L}$-structure $M$ is called **ultrahomogeneous**[a] iff any isomorphism $f\colon A \to A'$ with $A, A' \subseteq_{\text{fin}} M$ extends to an automorphism of $M$.

[a] sometimes also **homogeneous**, but not in this seminar

**Proposition 1.8.** Let $M$ be countable. Then $M$ has the EP iff $M$ is ultrahomogeneous.

*Proof.* “ $\Longrightarrow$ ” Take $M = M'$ in Proposition 1.6.

“ $\Longleftarrow$ ” Let $A$, $B$ be substructures of $M$ $f\colon A \to B$. Then $f\colon A \to f(A)$ is isomorphism, hence it extends to an isomorphism $h$ of $M$ and we can take

$$\begin{tikzcd} A \arrow[r, phantom, "\subseteq"] \arrow[d, "f"'] & M \\ B \arrow[ur, dashed, "h^{-1}"'] & \end{tikzcd}$$

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$\square$

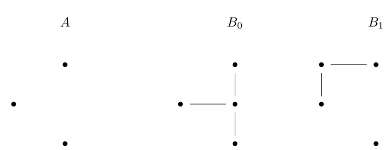
**Question 1.8.2.** Can we detect ultrahomogeneity from the age?

**Question 1.8.3.** When do we have these nice universal[a] objects, i.e. what classes of structures are ages of an ultrahomogeneous model?

[a]sometimes called generic

**Example 1.9.** Let $X$ be the class of finite forests.[a]

Suppose that $M$ is a countable ultrahomogeneous structure with age $k$.




We have $A \subseteq M$. $A \hookrightarrow B_0$ and $A \hookrightarrow B_1$. But now we get a contradiction, since simultaneously embedding $A$ into a copy $B_0$ and $B_1$, we get a circle.

[a]A **forest** is a graph without cycles. A **tree** is a connected forest.

**Definition 1.10.** A class $\mathcal{K}$ of finite $\mathcal{L}$-structures has the **amalgamation property** (**AP**) iff for all embeddings $f\colon A \to B_0$, $f_1\colon A \to B_1$ of finite structures in $\mathcal{K}$, there exists $C \in \mathcal{K}$ with embeddings $h_0\colon B_0 \to C$, $h_1\colon B_1 \to C$ such that $\forall a \in A.\ h_0 f_0(a) = h_1 f_1(a)$

$$
\begin{tikzcd}
 & B_0 \arrow[dr, dashed, "h_0"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & C \\
 & B_1 \arrow[ur, dashed, "h_1"'] &
\end{tikzcd}
$$

The AP is the real meat.

**Definition 1.11.** We say that $\mathcal{K}$ has **JEP** (**joint embedding property**) iff for all $A_0, A_1 \in \mathcal{K}$, there exists $B \in \mathcal{K}$ and embeddings $f_0 \colon A_0 \to B$, $f_1 \colon A_1 \to B$.

$$
\begin{tikzcd}
A_0 \arrow[dr, dashed, "f_0"] & \\
 & B \\
A_1 \arrow[ur, dashed, "f_1"'] &
\end{tikzcd}
$$

The JEP is a property we expect from every reasonable class.

**Proposition 1.12.** Let $M$ be a countable ultrahomogeneous structure. Then $\mathrm{Age}(M)$ has AP.

*Proof.* Consider

$$
\begin{tikzcd}
 & B_0 \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & \\
 & B_1
\end{tikzcd}
$$

We can assume $B_0, B_1 \subseteq M$. The isomorphism $f_0(a) \mapsto f_1(a)$ can be extended to $h \in \mathrm{Aut}(M)$. Then

missing picture

$$
\begin{tikzcd}
 & B_0 \arrow[dr, "h"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & h(B_0) \cup B_1 \\
 & B_1 \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}
$$

completes the diagram. $\square$

**Definition 1.13.** A hereditary[a] class $\mathcal{K}$ of finite $\mathcal{L}$-structures with the AP, JEP and countably many isomorphism types is called an **amalgamation class** (or **Fraïssé class**).

[a] closed under taking substructures, i.e. if $B \in \mathcal{K}$ and $A \hookrightarrow B$ then $A \in \mathcal{K}$

**Theorem 1.14** (Fraïssé's Theorem)**.** Let $\mathcal{K}$ be an amalgamation class. Then there exists a countable ultrahomogeneous $M$ with $\mathrm{Age}(M) = \mathcal{K}$.

**Definition 1.15.** Let $\mathcal{K}$ be an amalgamation class. An increasing chain $F_0 \subseteq F_1 \subseteq \ldots \subseteq$ of structures in $\mathcal{K}$ is called a **Fraïssé sequence** (or **rich sequence**) iff for each $F_i$ and for each $A \subseteq F_i$ and each embedding $f\colon A \to B$ with $B \in \mathcal{K}$, there exists $j > i$ and an embedding $h\colon B \to F_j$ such that the diagram commutes.

$$\begin{tikzcd}
A \arrow[r, phantom, "\subseteq"] \arrow[dr, "\forall f"'] & F_i \arrow[r, phantom, "\subseteq"] & F_j \\
& B \arrow[ur, dashed, "\exists h"'] &
\end{tikzcd}$$

**Lemma 1.16.** Let $\mathcal{K}$ be an amalgamation class. Let $(F_i)_i < \omega$ be a Fraïssé-sequence. Then $M = \bigcup_{i<\omega} F_i$ is ultrahomogeneous and $\mathrm{Age}(M) = \mathcal{K}$.

*Proof.* Let $A \in \mathcal{K}$. Use the JEP on $A$ and $F_0$ to get $B$. Since the $F_i$ form a Fraïssé-sequence, we get $B \hookrightarrow F_k$ for some $k$.

$$\begin{tikzcd}
F_0 \arrow[dr] & & F_j \\
& B \arrow[ur] & \\
A \arrow[ur] & &
\end{tikzcd}$$

$M$ has the EP:

$$\begin{tikzcd}
A \arrow[r, phantom, "\subseteq"] \arrow[dr] & F_i \arrow[r, phantom, "\subseteq"] & F_j \\
& B \arrow[ur, dashed] &
\end{tikzcd}$$

$\square$

*Proof of Theorem 1.14.* By Lemma 1.16 it suffices to build a Fraïssé-sequence. Let $\sigma\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection such that if $\sigma(n) = (i, j)$, then $n \geqslant i$. This is called a "scheduling function".

Take $F_0 \in \mathcal{K}$. List all embeddings $f_{0,0}, f_{0,1}, \ldots$ of finite substructures of $F_0$ into other structures of $\mathcal{K}$ (up to isomorphism).

At stage $F_k$, deal with the embedding $f_{\sigma(k)}$ using AP.

$$\begin{tikzcd} A \arrow[r, phantom, "\subseteq"] \arrow[dr, "f_{0,0}"'] & F_0 \arrow[r, phantom, "\subseteq"] & F_1 \\ & B \arrow[ur] & \end{tikzcd}$$

$\square$

**Example 1.17.** Consider the class of finite graphs. This has AP: Let $A \hookrightarrow B_0$, $A \hookrightarrow B_1$. Take the free amalgam of $B_0$ and $B_1$ along $A$.

So by Fraïssé's Theorem (1.14) we get the random graph.

**Example 1.18.** Consider the class of triangle-free finite graphs. Note that the free amalgam does not create triangles, so Fraïssé's Theorem (1.14) can be applied. The resulting graph is called the **Henson graph** $H_3$ (or "**triangle-free random graph**"[a]).

[a] this is a bad name

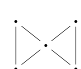
For the Henson graph the analogous statement to the witness property is:

$\forall U, V$ finite and disjoint and $U$ edge free, there exists $w$ such that ...

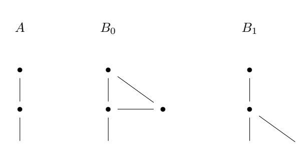
The same thing can also be done the obtain the $K_n$-free random graph.

**Example 1.19.** Non-examples are:

- finite forests
- finite bowtie-free graphs, i.e. not containing



  as a graph-theoretic subgraph. Consider



**Theorem 1.20** (Lachlan-Woodrow)**.** Any ultrahomogeneous countably infinite graph is

- the random graph $R$,
- $H_n$, the $K_n$-free graph or
- the disjoint union of $K_n$s (where maybe $n = \infty$)

or its complement.

We will not prove this in this seminar.

## 2 More Fraïssé: algebraic structures, other examples

2[nd] talk, 2024-04-16, Zahra Mohammadi

TODO

## 3 $\omega$-Categoricity and more Model Theory

3[rd] talk, 2024-04-30, Rob Sullivan

We again consider relational structures.

**Definition 3.1.** A Fraïsse structure $M$ has the **strong EP** if

$$\forall A \overset{\text{finite}}{\subseteq} M.$$
$$\forall A \xrightarrow{f} B \text{ with } B \in \operatorname{Age}(M).$$
$$\forall V \overset{\text{finite}}{\subseteq} M \text{ disjoint from A.}$$
$$\exists \text{ embedding } h\colon B \to M$$
$$\text{such that}$$
$$\begin{tikzcd} A \arrow[rr, phantom, "\subseteq"] \arrow[dr, "f"'] & & M \\ & B \arrow[ur, "h"'] & \end{tikzcd}$$
$$\text{and } h \text{ avoids } V \text{ (i.e. } \operatorname{im} h \cap V = \varnothing).$$

**Lemma 3.2.** Let $M$ be a Fraïsse structure. Then $M$ has the strong EP iff $\operatorname{Age}(M)$ has the strong AP.

*Proof.* $\square$

TODO

**Lemma 3.3.** Let $M$ be a Fraïsse structure. Then $\mathrm{Age}(M)$ has the strong AP iff $\forall V \overset{\text{finite}}{\subseteq} M$. $M \setminus V \cong M$.

*Proof.* $\Longrightarrow$ Take $A \in \mathrm{Age}(M)$. Use the strong EP over $\varnothing$ avoiding $V$ to get a copy of $A$ inside $M \setminus V$. So $\mathrm{Age}(M \setminus V) = \mathrm{Age}(M)$. The strong EP for $M$ implies that $M \setminus V$ has EP. Use Fraïssé's theorem.

$\Longleftarrow$ The condition immediately implies strong EP. □

**Example 3.4.** Let $M$ be the disjoint union of two copies of the random graph. For all $V \overset{\text{finite}}{\subseteq} M$, we have $M \setminus V \cong M$. However $M$ is not ultrahomogeneous.

**Lemma 3.5** (Neumann)**.** Let $G \curvearrowright \Omega$ be a group action without finite orbits.

Then for all $A, B \overset{\text{finite}}{\subseteq} \Omega$, there exists $g \in G$ such that $gA \cap B = \varnothing$.

*Proof.* If $|A| = 1$, this is trivial.

Since there are no finite orbits, for all $A \overset{\text{finite}}{\subseteq} \Omega$ the set $\{gA : g \in G\}$ is infinite.

Fix $B$ and do induction on $|A|$.

We'll only look at some finite cases. Let $|A| = 2$. Suppose that every translate of $A$ intersects $B$. For $x \in \Omega$, let $T_x$ be the set of translates of $A$ containing $x$.

Let $b_0 \in B$. It suffices to show that $T_{b_0}$ is finite (since then it follows that the set of translate of $A$ is finite, which is a contradiction).

By the case of $|A| = 1$, there there exists $g_0 \in G$ such that $g_0 b_0 \notin B$. For $gA \in T_{g_0 b_0}$, we have $gA \cap b \neq \varnothing$, so as $|A| = 2$ and $B$ is finite, so $T_{g_0 b_0}$ is finite, hence $T_{b_0}$ is finite.

Let $|A| = 3$ Let $b_0 \in B$, Then there exists $b_0 \in G$ such that $g_0 b_0 \notin B$. Each $gA \in T_{g_0 b_0}$ has $gA \cap B \neq \varnothing$. So $T_{g_0 b_0} = \bigcup_{b_1 \in B}, T_{g_0 b_0, b_1}$. From the case of $|A| = 2$ , there exists $g_1$ with $g_1 g_0 b_0, g_1 b_1 \notin B$. It follows that $T_{g_1 g_0 b_0, g_1 b_1}$ is finite.

The general case is left as an exercise. □

**Definition 3.6.** Let $G \curvearrowright \Omega$ be a group action and $A \overset{\text{finite}}{\subseteq} \Omega$. The **finite-orbit closure**, $\mathrm{fcl}(A)$, is the union of the finite orbits of the pointwise stabilizers $G_{(A)}$ of $A$.

Note that $A \subseteq \mathrm{fcl}(A)$. We say that $\mathrm{fcl}(A)$ is trivial iff $\mathrm{fcl}(A) = A$.

**Proposition 3.7.** Let $M$ be a Fraïssé-structure. Then $M$ has strong amalgamation iff $\forall A \overset{\text{finite}}{\subseteq} M, \mathrm{fcl}(A) = A$.

*Proof.* " $\implies$ " Take $m \in M \setminus A$. Let $k \geqslant 1$. Take the strong amalgam of $k$ instances of $A \hookrightarrow A \cup \{m\}$. Then use the EP over $A$. So $\mathrm{Orb}_{G_{(A)}}(M)$ is infinite.

" $\impliedby$ "

Work inside $M$. We can assume $A \subseteq B_0 \cap B_1 \subseteq M$. Consider $G_{(A)} \curvearrowright M \setminus A$ and apply Neumann's Lemma (3.5). $\square$

Let's get formulae involved.

**Question 3.7.4.** Recall $\mathcal{L}_{\text{graph}} = \{\sim\}$. What graph-theoretic properties are axiomatizable in first-order logic? I.e. given a graph property $P$, is there a theory $T$ such that $\Gamma \models T \iff \Gamma$ has property $P$ for all graphs $\Gamma$.

- Being bipartite is axiomatizable, since a graph is bipartite iff it does not contain a circle of odd diameter.

  (Note that we can not directly talk about partitions in first-order formulae.)

- Not being bipartite is not aximoatizable, since being bipartite but not containing a cycle of length $\leqslant n$ is consistent.

- Connectedness and not-connectedness are not axiomatizable:

  Let $T$ be the theory saying that every vertex has degree 2 and the graph is acyclic. $T$ has no finite models. If $\Gamma \models T$, then $\Gamma$ consists of disjoint copies of $\mathbb{Z}$. So $T$ is $\kappa$-categorical for $\kappa$ uncountable. Hence $T$ is complete by Vaught's Test (3.9). In particular, $\mathrm{Th}(\mathbb{Z}) = \mathrm{Th}(\mathbb{Z} \sqcup \mathbb{Z})$.

**Definition 3.8.** Let $\mathcal{L}$ be a language (not necessarily relational). Let $\kappa \geqslant |\mathcal{L}|$ be a cardinal. Let $T$ be an $\mathcal{L}$-theory with infinite models. Then $T$ is **$\kappa$-categorical** iff all models of $T$ of cardinality $\kappa$ are isomorphic.

**Theorem 3.9** (Vaught's test)**.** Let $T$ be a consistent $\kappa$-categorical theory with no finite models. Then $T$ is complete.

*Proof.* Let $M, N$ be models of $T$. Using Löwenheim-Skolem, we can find $M', N'$ with $\mathrm{Th}(M') = \mathrm{Th}(M)$ and $\mathrm{Th}(N') = \mathrm{Th}(N)$ such that $|M'| = |N'| = \kappa$. Then $M' \cong N'$, hence $M \equiv N$. $\square$

**Theorem 3.10** (Tarski's test)**.** If $\mathcal{B}$ is an $\mathcal{L}$-structure with domain $B$ and $A \subseteq B$. Then $A$ is the domain of an elementary substructure of $B$ iff for every formula $\varphi(x)$ with $\mathcal{B} \models \varphi(b)$ for some $b \in B$, we have $\mathcal{B} \models \varphi(a)$ for some $a \in A$.

**Definition 3.11.** Let $\mathcal{L}$ be a language (not necessarily relational). Let $T$ be an $\mathcal{L}$-theory. Let $\Sigma(x)$ be a set of forumlae. We say that $\Sigma(x)$ is **isolated mod $T$** iff three exists an $\mathcal{L}$-formula $\varphi(x)$ such that

- $T \cup \{\varphi(x)\}$ is consistent.
- For $\sigma(x) \in \Sigma(x)$, $T \models \forall x.(\varphi(x) \to \sigma(x))$.

**Example 3.12.** Let $T$ be a complete theory. Assume that $\Sigma(x)$ is isolated mod $T$. Then every model of $T$ realizes $\Sigma(x)$, i.e. $(\exists x)\varphi(x) \in T$.

A key tool for Theorem 4.12 is:

**Theorem 3.13** (Omitting types theorem)**.** Let $\mathcal{L}$ be countable. Let $T$ be a consistent theory. Let $\Sigma(x)$ be a set of formulae not isolated mod $T$. Then $T$ has a model not realising ("**omitting**") $\Sigma(x)$.

*Proof.* Pick a countably infinite set $\mathcal{C}$ of new constants. We'll extend $T$ to an $\mathcal{L}(\mathcal{C})$-theory $T^*$ such that

(a) $T^*$ is a Henkin theory, i.e. for every $\mathcal{L}(\mathcal{C})$ formula $\varphi(x)$, there is $c \in \mathcal{C}$ such that $(\exists x.\ \varphi(x)) \to \varphi(c)) \in T^*$.

(b) For all $c \in \mathcal{C}$, there is $\sigma(x) \in \Sigma(x)$ with $\neg\sigma(c) \in T^*$.

We construct $T^*$ inductively as the union of an increasing chain of consistent theories $T = T_0 \subseteq T_1 \subseteq \dots$. At each stage $T_n \subseteq T_{n+1}$ we add finitely many $\mathcal{L}(\mathcal{C})$-sentences. Enumerate $\mathcal{C} = \{c_i : i < \omega\}$ and enumerate all $\mathcal{L}(\mathcal{C})$-formulae in $x$ as $\{\psi_i(x) : i\omega\}$.

Given $T_{2i}$, take a constant $c \in \mathcal{C}$ not occurring in $T_{2i} \cup \{\psi_i(x)\}$. Let $T_{2i+1} :=$ $T_{2i} \cup \{(\exists x.\ \psi_i(x)) \to \psi_i(c)\}$.

Given $T_{2i+1} = T \cup T$, let $\delta(c_i, \overline{c}) = \bigwedge F$, $\overline{c} \subseteq \mathcal{C}$ $\overline{c} \not\ni c_i$. Then $\delta(x, \overline{y})$ is an $\mathcal{L}(\mathcal{C})$-formula and

$$\exists \overline{y} d(x, \overline{y})$$

is consistent with $T$, but it does not isolate $\Sigma(x)$.

So there exists $\sigma(x) \in \Sigma(x)$ and a model $M \models T$ with $M \models (\exists x) . (\exists y.\ \delta(x, \overline{y}) \wedge \neg\sigma(x))$.

Let $T_{2i+2}. = T_{2i+1} \cup \{\neg\sigma(c_i)\}$. Then $T_{2i+2}$ is consistent.

Let $(\mathcal{B}, (a_c)_{c \in \mathcal{C}}) \models T^*$.

Let $A := \{a_c : c \in \mathcal{C}\}$. By Tarski's Test (3.10) and since $T^*$ is Henkin, this is an elementary substructure of $\mathcal{B}$, □

# 4 $\omega$-Categoricity and Ryll-Nardzewski

4th talk, Prof. Hils, 2024-05-07

**Question 4.0.5.** What is special about $\aleph_0$ with respect to categoricity?[a]

[a] The name $\omega$-categoricity is used for historical reasons, but of course "$\aleph_0$-categoricity" would be a better name.

Motivation:

**Example 4.1.** Let $\mathcal{L} = \{f, P\}$, where $f$ is a unary function and $P$ a unary predicate. Let $T$ be the theory saying that models are infinite, $f \circ f = \mathrm{id}$ and $\forall x.\ x \in P \iff f(x) \notin P$.

This is $\kappa$-categorical for every $\kappa \geqslant \aleph_0$. Let $\mathcal{L}' := \{P\}$. Let $T' := T|_{\mathcal{L}'}$.

$T'$ is $\omega$-categorical, but not $\kappa$-categorical for $\kappa > \aleph_0$.

Note that $\omega$-categoricity passes to reducts, but this is not the case for $\kappa > \omega$.

**Definition 4.2.** A **complete $n$-type** in $T$, $p(x_1, \ldots, x_n)$ is a maximal finitely satisfiable set of $\mathcal{L}$-formulas $\varphi(x_1, \ldots, x_n)$.

Let $S_n(T)$ be the set of all complete $n$-types.

Let $A \subseteq \mathcal{M} \models T$, $T_A := \mathrm{Th}(\mathcal{M}_A)$. Let $S_n(T_A) =: S_n(A)$.

(With this notation we also write $S_n(\varnothing) := S_n(T)$).

**Example 4.3.** For $\overline{b} \in \mathcal{M}^n$ and $A \subseteq \mathcal{M}$, we set

$$\mathrm{tp}^m(\overline{b}/A) := \{\varphi(\overline{x})\ \mathcal{L}_A\text{-formula} : S_n(A) \ni M \models \varphi[\overline{b}]\}.$$

Any $p \in S_n(A)$ is of this form: By compactness, there exists $\mathcal{N} \succeq \mathcal{M}$, such that $\exists \overline{b} \in \mathcal{N}^n.\ \mathrm{tp}(\overline{b}/A) = p$.

**Definition 4.4.** The **stone topology** on $S_n(T)$ is the topology with basis of open sets given by $\langle\varphi\rangle \subseteq S_n(T)$, for $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula and $\langle\varphi\rangle := \{p : \varphi \in p\}$.

**Fact 4.4.6.** $S_n(T)$ is compact, Hausdorff and totally disconnected. (It even is profinite)

**Remark 4.4.7.**

(1) $p \in S_n(T)$ is **isolated** iff there exists $\varphi$, such that $\langle \varphi \rangle = \{p\}$. This is equivalent to $p$ being (topologically) isolated in $S_n(T)$.

(2) Exercise: If $\overline{a}, \overline{b}$ are finite tuples from $\mathcal{M} \models T$, then $\mathrm{tp}(\overline{a}, \overline{b})$ is isolated iff $\mathrm{tp}(\overline{b})$ and $\mathrm{tp}(\overline{a}/\overline{b})$ are both isolated.

**Definition 4.5.**

- $\mathcal{M}$ is **atomic** iff it only realizes isolated $n$-types for all $n \geqslant 0$.
- $\mathcal{M}$ is **$\omega$-saturated** iff for all $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ all $p \in S_n(A)$ are realized in $\mathcal{M}$.

  (equivalently this holds for all $p \in S_1(A)$).

**Fact 4.5.8.** For every $\mathcal{M} \models T$ there exists an elementary extension $\mathcal{N} \succeq \mathcal{M}$ which is $\omega$-saturated.

*Proof.* Catch your own tail: Take $\mathcal{M}_1 \succeq \mathcal{M}$ realizing all $p \in S_1(A)$ for all $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ (this exists by compactness) and similarly $\mathcal{M}_{n+1} \succeq \mathcal{M}_n$.

Let $\mathcal{N} := \bigcup_{i<\omega} \mathcal{M}_i$. □

**Definition 4.6.** If

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \cap\!\!| & & \cap\!\!| \\ \mathcal{M} & & \mathcal{M}' \end{array}$$

where $\mathcal{M}, \mathcal{M}' \models T$, $f$ is called a **partial elementary** embedding if it preserves formulas with variables form $A$, i.e.

$$\mathcal{M} \models \varphi(\overline{a}) \iff \mathcal{M}' \models \varphi(f(\overline{a}))$$

for all $\overline{a} \in A^n$, i.e. $\mathrm{tp}(A) = \mathrm{tp}(f(A)) = \mathrm{tp}(A')$.

**Definition 4.7.** $\mathcal{M}$ countable is called **homogeneous**iff every partial elementary $f \colon A \to B$, $A, B \overset{\text{finite}}{\subseteq} \mathcal{M}$ extends to $\sigma \in \mathrm{Aut}(\mathcal{M})$.

**Proposition 4.8.**

(1) Let $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M}$ countable atomic.

Then any $f\colon A \to \mathcal{N}$ partial elementary with $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ extends to an elementary embedding $F : \mathcal{M} \overset{\preceq}{\hookrightarrow} \mathcal{N}$.

(2) If $\mathcal{M}$ and $\mathcal{N}$ are both atomic then any $f$ as in (1) extends to an isomorphism. In fact the atomic countable modle is unique if it exists.

(3) $\mathcal{M}$ countable atomic $\implies$ $\mathcal{M}$ homogeneous.

*Proof.* Let $f\colon A \to \mathcal{N}$. Let $\mathcal{M} = \{a_i | i \in \mathbb{N}\}$. Given $f_k : \{a_0, \ldots, a_h\} \to \mathcal{N}$ partial elementary. Consider $\operatorname{tp}(a_{h+1}/a_{\leqslant h}))$ this is isolated by assumption. Hence $f_{h*}$ is an isolated type in $S_1(\operatorname{im}(f_h))$. Let $b_{h+1}$ be a realization, $a_{h+1} \mapsto b_{h+1}$.

(2) is a symmetric version of this and (3) is a special case of (2). ☐

**Proposition 4.9.**

(1) Let $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M}$ countable, $\mathcal{N}$ $\omega$-saturated. Then every $f\colon A \to \mathcal{N}$ partial elementary, where $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ extends to $\tilde{f}\colon \mathcal{M} \overset{\preceq}{\hookrightarrow} \mathcal{N}$.

(2) Let $\mathcal{M}, \mathcal{N}$ be countable and $\omega$-saturated. Then every $f$ as in (1) extends to an isomorphism.

In particular, a countable $\omega$-saturated model is unique (if it exists).

(3) If $\mathcal{M}$ is countable and $\omega$-saturated, then $\mathcal{M}$ is homogeneous.

*Proof.* (1) is basically the same as before using $\omega$-saturation of $\mathcal{N}$.

Again (2) is a symmetric version and (3) a special case. ☐

**Example 4.10.**

- Consider $\mathrm{ACF}_0$. $\mathbb{Q}^{\mathrm{alg}}$ is atomic. Let $\{t_i | i \in \mathbb{N}\}$ be algebraically independent. Then $\mathbb{Q}(t_i | i \in \mathbb{N})$ is $\omega$-saturated.
- Consider infinite $\mathbb{Q}$-vector spaces. $\mathbb{Q}$ is atomic. $\mathbb{Q}^{(\mathbb{N})}$ is $\omega$-saturated.
- Graphs: $\mathbb{Z}$ is atomic, $\coprod_{i<\omega} \mathbb{Z}$ is $\omega$-saturated.
- DOAG[a] $= \operatorname{Th}(\mathbb{Q}, +, -, \leqslant, 0)$: $\mathbb{Q}$ is atomic. However no $\omega$-saturated countable model exists, since $|S_2(T)| = 2^{\aleph_0}$ (reals can be coded in 2-types).

[a] divisible ordered abelian groups

**Observe.** *If $G \curvearrowright X$, we get $G \curvearrowright X^n$ via $g.(x_1, \ldots, x_n) \coloneqq (gx_1, \ldots, gx_n)$.*

**Definition 4.11.** A group action $G \curvearrowright X$ is called **oligomorphic** if for every $n$, $G \curvearrowright X^n$ has finitely many orbits.

**Observe.**[5] *$X \subseteq \mathcal{M}^n$ is* $\operatorname{Aut}(\mathcal{M})$*-invariant iff it is the union of* $\operatorname{Aut}(\mathcal{M})$*-orbits.*

**Theorem 4.12** (Ryll-Nardzewski, Engler, Svenonius)**.** Let $T$, $\mathcal{L}$ as above.

The following are equivalent:

(1) $T$ is $\omega$-categorical.

(2) Any $p \in S_n(T)$ is isolated for all $n \in \mathbb{N}$.

(3) $S_n(T)$ is finite for all $n$.

(4) There are infinitely many $\mathcal{L}$-formulae $\varphi(x_1, \ldots, x_n)$ up to $\sim_T$.

(5) There exists $\mathcal{M} \models T$ countable, such that $\operatorname{Aut}(\mathcal{M}) \curvearrowright \mathcal{M}$ is oligomorphic.

(6) There exists $\mathcal{M} \models T$ countable such that all $\operatorname{Aut}(\mathcal{M})$-invariant $X \subseteq \mathcal{M}^n$ are $\varnothing$-definable (i.e. $X = \varphi[M]$).

(7) There exists $\mathcal{M} \models T$ countable such that $\mathcal{M}$ only realizes finitely many $n$-types for all $n \in \mathbb{N}$.

*Proof.* (1) $\implies$ (2): Suppose not, i.e. $p \in S_n(T)$ is not isolated. Then there exists a countable model $\mathcal{M} \models T$ realizing $p$ (Löwenheim-Skolem, compactness). By the Omitting Types Theorem (3.13), there exists $\mathcal{N} \models T$ countable omitting $p$. $\mathcal{M} \cong \mathcal{N}$ $\lightning$.

(2) $\iff$ (3) $\iff$ (4) are obviously equivalent.

Assume (2). For $\mathcal{M} \models T$ countable, we then have $\operatorname{tp}(\overline{a}) = \operatorname{tp}(\overline{b})$, $\overline{a}, \overline{b} \in \mathcal{M}^n$ iff there exists $\sigma \in \operatorname{Aut}(\mathcal{M})$ such that $\sigma(\overline{a}) = \overline{b}$. (here we use that $\mathcal{M}$ is atomic and hence homogeneous). So $\operatorname{Aut}(\mathcal{M})$-orbits in $\mathcal{M}^n$ correspond to $n$-types realized in $\mathcal{M}$. We get (2) $\implies$ (5): By (2) $\implies$ (3), we get that $S_n(T)$ is finite.

(2) $\implies$ (6): By (2) $\implies$ (3), every Aut-invariant subset $X \subseteq \mathcal{M}^n$ is a finite union of realizations of isolated types.

(6) $\implies$ (7): Let $\mathcal{M}$ be as in (6). Suppose $\mathcal{N}$ realizes infinitely many $n$-types $p_1, p_2, \ldots$ For $I \subseteq \mathcal{N}$ let $Y_I := \bigcup_{i \in I} p_i[\mathcal{M}]$. This gives rise to $2^{\aleph_0}$ invariant subsets. But $\mathcal{L}$ is countable.

(7) $\implies$ (3):

Set set of types $p \in S_n(T)$ realized in $\mathcal{M}$ is dense in $S_n(T)$. So everything, by finiteness. (5) $\implies$ (7): trivial.

(2) $\implies$ (1): Countable atomic models are unique by the proposition. $\square$

[5] or use this as a definition.

**Example 4.13.** (1) Let $\mathcal{L}_{\text{gp}} = \{e, \circ\}$ be the language of groups and $G$ infinite and $\omega$-categorical. Then there exists $n$ such that $g^n = e$ for all $g \in G$, i.e. $G$ is of finite exponent.

Indeed, there can be only finitely many finite orders. There exists no element $g$ of infinite order, as otherwise, $(g, g^n)$ would be $n$ distinct 2-types.

(2) No infinite field is $\omega$-categorical.

By (1), characteristic 0 is impossible. If $K$ is infinite, then then for all $n \in \mathbb{N}$, there exists $x \in K^*$ with $x^n \neq 1$.

(3) If $\mathcal{L}$ is a finite language without function symbols, and $T$ a complete $\mathcal{L}$-theory with infinite models and quantifier elimination, then $T$ is $\omega$-categorical.

E.g. an ultrahomogeneous structure in a finite language without function symbols. E.g. $\mathrm{Fr}(K)$, where $K$ is a Fraïssé class in $\mathcal{L}$ as before.

**Corollary 4.14.** If $T$ is an $\omega$-categorical $\mathcal{L}$-theory and $\mathcal{L}' \subseteq \mathcal{L}$, then $T' := T|_{\mathcal{L}'}$ is $\omega$-categorical.

# 5 Ryll-Nardzewski II

$5^{\text{th}}$ talk, Martin Hils, 2024-05-14

Let $\mathcal{L}$ be countable.

Recall that if $\mathcal{M}$ is a countable, ultrahomogeneous $\mathcal{L}$-structure in a finite language without function symbols, then $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical.

Indeed, $\mathrm{Aut}(\mathcal{M}) \rightharpoonup \mathcal{M}$ is oligomorphic in this case.

Now we want to consider languages that also have function symbols:

**Definition 5.1.** Let $\mathcal{L}$ be finite (possibly including function symbols). A class $\mathcal{K}$ of finitely generated $\mathcal{L}$-structurs is called **uniformly locally finite** if there exists $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $A \in \mathcal{K}$ generated by $n$ elements we have $|A| \leqslant f(n)$.

An infinite $\mathcal{L}$-structure $\mathcal{M}$ is called **uniformly locally finite** if $\mathrm{Age}(\mathcal{M})$ is.

Recall from last week: If $G$ is an $\omega$-categorical group, then $G$ is of finite exponent. By the same argument we get the following:

**Lemma 5.2.** Let $\mathcal{L}$ be a countable language. If $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical, then $\mathcal{M}$ is uniformly locally finite.

Recall:

**Definition 5.3.** If $A \subseteq \mathcal{M}$, then the model-theoretic algebraic closure is defined

$$\operatorname{acl}(\mathcal{M}) = \bigcup_{\substack{\varphi \mathcal{L}_A\text{-formula} \\ \varphi[M] \text{ finite}}} \varphi[M].$$

We have $A \subseteq \langle A \rangle \subseteq \operatorname{dcl}(A) \subseteq \operatorname{acl}(A) = \operatorname{acl}(\operatorname{acl}(A))$.

**Example 5.4.** In vector spaces, this is the linear span. In ACF it is the field theoretic algebraic closure, hence the name.

**Definition 5.5.** $\mathcal{M}$ is said to have **uniformly finite algebraic closure** iff there exists $g\colon \mathbb{N} \to \mathbb{N}$ such that for all $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ we have $|\operatorname{acl}(A)| \leqslant g(|A|)$.

**Lemma 5.6** (Lemma 5.2'). If $\mathcal{M}$ is $\omega$-categorical, then $\mathcal{M}$ has uniformly finite algebraic closure.

*Proof.* If $\operatorname{tp}(a_1, \ldots, a_n) = \operatorname{tp}(b_1, \ldots, b_n)$, then $\operatorname{acl}(a_1, \ldots, a_n) \cong \operatorname{acl}(b_1, \ldots, b_n)$.

Hence uniform finiteness of acl follows from finiteness of $\operatorname{acl}(a_1, \ldots, a_n)$. Towards a contradiction assume that $\operatorname{acl}(a_1, \ldots, a_n) =: \tilde{A}$ is infinite, say $\tilde{A} = (b_i)_{i \in \mathbb{N}}$. If $\operatorname{tp}(a_1, \ldots, a_n, b_j) = \operatorname{tp}(a_1, \ldots, a_n, b_k)$, there exists an $\mathcal{L}_A$-formula $\varphi(x)$ with $\varphi[M]$ finite, such that $b_j, b_k \in \varphi[M]$. Hence there are infinitely many different types of the form $\operatorname{tp}(a_1, \ldots, a_n, b_j)$ $\lightning$. ☐

Recall:

**Definition 5.7.** $T$ has **quantifier elimination** (QE) if every $\mathcal{L}$-formula $\varphi(X_1, \ldots, X_n)$ is equivalent modulo $T$ to a quantifier free formula.

**Proposition 5.8.** Suppose that $\mathcal{L}$ is finite an $\mathcal{M}$ a countably infinite $\mathcal{L}$-structure. Then the following are equivalent:

(1) $\mathcal{M}$ is ultrahomogeneous and uniformly locally finite.

(2) $\mathcal{M}$ is $\omega$-categorical and has QE.

*Proof.* (1) $\implies$ (2): Since there are only finitely many $n$-generated substructures, there are only finitely many orbits of $\operatorname{Aut}(\mathcal{M}) \rightharpoonup \mathcal{M}$ by ultrahomogeneity.

Since $\operatorname{tp}(\overline{a}) = \operatorname{tp}(\overline{b}) \iff \langle \overline{a} \rangle = \langle \overline{b} \rangle$, we get that $\{\operatorname{tp}(\overline{a})\} = \langle \varphi \rangle$ for some quantifier free formula $\varphi$.

Every $\mathcal{L}$-formula is equivalent modulo $T$ to a finite disjunction of such formulae.

Alternative proof of QE:

**Fact 5.8.9.** Let $T$ be a theory such that for all $\mathcal{M}, \mathcal{N} \models T$, $A \subseteq \mathcal{M}, \mathcal{N}$ a finitely generated common substructure, $\mathcal{N}$ $\omega$-saturated, $b \in \mathcal{M}$, there is $b' \in \mathcal{N}$ such that $\langle A, b\rangle = \langle A, b'\rangle$ via $b \mapsto b'$.

(2) $\implies$ (1): We already know uniformly locally finite by Lemma 5.2 (this does not need QE).

As $\mathcal{M}$ is $\omega$-saturated (by $\omega$-categoricity) and we have QE, it follows that we have the extension property. This implies ultrahomogeneity. ☐

*Proof (Alternative Proof).* Let $A$ be a finite $\mathcal{L}$-structure generated by $(a_1, \ldots, a_n)$. Then there is a quantifier free formula $\psi_{A,\overline{a}}(x_1, \ldots, x_n)$ (given by the simple diagram of $A$) such that for all $\overline{b} = (b_1, \ldots, b_n)$, $\overline{b} \in \mathcal{N}$, then $\mathcal{N} \models \psi_{A,\overline{a}}(\overline{b})$ iff $\overline{b} \mapsto \overline{a}$ defines an isomorphism $\langle\overline{b}\rangle \cong \langle\overline{a}\rangle$. Now if $\mathrm{Age}(\mathcal{M})$ is uniformly locally finite, there are only finitely many substructures of $\mathcal{M}$ generated by $n$ elements up to isomorphism.

- Let $U_0$ be the following set of $\mathcal{L}$-sentences:

  $$\forall \overline{x}.\ (\psi_{A,\overline{a}}(\overline{x}) \to \exists y.\ \psi_{B,\overline{a}^\frown b}(\overline{x}, y)),$$

  where $A = \langle a_1, \ldots, a_n\rangle \subseteq B = \langle a_1, \ldots, a_n, b\rangle$. For $n = 0$ take $\exists y.\ \psi_{B,b}(y)$.

- Let $U_1$ be the following set of sentences:

  $$\forall x_1, \ldots, x_n.\ \bigvee_{\substack{A,\overline{a}\\ |\overline{a}|=n}} \psi_{A,\overline{a}}(x_1, \ldots, x_n).$$

If $\mathcal{N} \models U_0 \cup U_1$ is countably infinite, then $\mathcal{N} \cong \mathcal{M}$ as $\mathcal{N}$ is the Fraïssé limit of $\mathrm{Age}(\mathcal{M}) = \mathrm{Age}(\mathcal{N})$.

☐

**Example 5.9.**

- Let $K$ be the class of all finite groups. $K$ is locally finite, but not uniformly locally finite. $K$ is a Fraïsse class (cf. section 2) $\mathrm{Fr}(K)$ is not $\omega$-categorical (since it is not of finite exponent). (it also does not have QE, but we don't want to check this).
- Let $K$ be the class of all finite abelian groups. Then $\mathrm{Fr}(K) = \bigoplus_{i\in\mathbb{N}}(\mathbb{Q}/\mathbb{Z})$. $\mathrm{Th}(\mathrm{Fr}(K))$ has QE (since it is divisible).

  We have $\bigoplus_{n\in\mathbb{N}} \mathbb{Q}/\mathbb{Z} \preceq \bigoplus_{n\in\mathbb{N}} \mathbb{Q}/\mathbb{Z} \oplus \bigotimes_{j\in J} \mathbb{Q}$, so it is not $\omega$-categorical. (We already knew this since it is not uniformly locally finite.)
- Fix $n \geqslant 2$. Let $K_n$ be the class of all finite abelian groups of expo-

nent $n$. Exercise (easy): $K_n$ is a Fraïsse class. We have $\mathrm{Fr}(K_n) \cong \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/n =: A_n$.

This is $\omega$-categorical, but not always $\aleph_1$-categorical. It is $\aleph_1$-categorical iff $n = p^m$ for some prime $p$.

Let $n = p_1 \cdot \ldots \cdot p_l$. We have $A_n \geqslant p_1 A_n \geqslant p_1 p_2 A_n \geqslant \ldots \geqslant 0$. Hence $\mathrm{MR}\,(\bigoplus_{i\in\mathcal{N}} \mathbb{Z}/n\mathbb{Z}) \geqslant l$ (in fact equality holds).

- Generic BLF: a linear algebra "analogue" of the random graph.

  Fix a prime $p$. Consider the class $K$ of $(V, \mathbb{F}_p, \beta)$, where $V$ is a finite dimensional $\mathbb{F}_p$-vector space, $\beta\colon V^2 \to \mathbb{F}_p$ is a symmetric bilinear form.

  This is a Fraïssé class. JEP and HP are clear.

  Consider $\mathrm{Fr}(K) =: (V, \mathbb{F}_p, \beta) =: \mathcal{M}$. We know that $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical with QE.

  It satisfies the following axioms:

  – Symmetric bilinear form on $\mathbb{F}_p$-vector spaces.

  – Given $\lambda_1, \ldots, \lambda_n \in \mathbb{F}_p$, for all $x_1, \ldots, x_n \in V$, iff $x_1, \ldots, x_n$ are $\mathbb{F}_p$-linearly independent, then there exists $y$ such that $\bigwedge_{i=1}^n \beta(x_i, y) = \lambda_i$

  This already is a full axiomatization.

  Random graph and generic BLF have IP.

**Proposition 5.10** (Proposition 5.8'). Let $\mathcal{L}$ be finite and $\mathcal{M}$ countably infinite.

Then the following are equivalent:

(1) $\mathcal{M}$ has uniformly finite acl and any isomorphism between finite algebraically closed subsets of $\mathcal{M}$ extends to an automorphism $\sigma \in \mathrm{Aut}(\mathcal{M})$.

(2) $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical and every $\mathcal{L}$-formula is equivalent modulo $T$ to a boolean combination of bounded existential formulae. , where bounded existential means $\varphi(\overline{x}) := \exists \overline{y}.\ \psi(\overline{x}, \overline{y}$, where $\psi$ is quantifier free and such that $\mathcal{M} \models \psi(\overline{a}, \overline{b})$, then $\overline{b} \in \mathrm{acl}(\overline{a})$.

*Proof.* (2) $\implies$ (1) by Lemma 5.6 and the fact that the formulae of the form in question determine the complete type.

(1) $\implies$ (2): Write down axioms for the extension property of finite acl-closed sets. ☐

## 6 The Random Graph and Ehrenfeucht-Fraïssé Games

$6^{\text{th}}$ talk, Shujie Yang, 2024-05-28

Recall the witness property: Given finitely many distinct vertices $u_1, \ldots, u_m$ $v_1, \ldots, v_n$ there exist a vertex $z$ which is adjacent to all of the $u_i$ and none of the $v_j$. We called this $z$ **correctly joint** with $\overline{u}, \overline{v}$.

Let $R$ denote the random graph.

**Proposition 6.1.** The result of any of the following operations $R$ is isomorphic to $R$:

(1) Deleting a finite number of vertices.

(2) Deleting or adding a finite number of edges.

(3) Switching with respect to a finite set of vertices, i.e. remove all outgoing edges and add all outgoing non-edges.

**Fact 6.1.10.** $R$ satisfies the following pigeonhole principle: Partitioning the vertices of $R$ into finitely many groups, at least one of the induced subgraphs is isomorphic to $R$.

*Proof.* Let $R = X_1 \cup X_2 \cup \ldots \cup X_n$. Suppose none of the $X_i$ is isomorphic to $R$. Let $u_i, v_i \in X_i$ be such that there is no correctly joined $z$.

However let $U = \bigcup_i u_i$, $V = \bigcup_i v_i$. Then there must be $z$ in $R$ correctly joined to $U, V$ and $z \in X_i$ for some $i$ $\lightning$. □

**Proposition 6.2.** The only countable graphs $\Gamma$ with satisfy this pigeonhole principle are

- the complete graph,
- the null graph and
- $R$.

*Proof.* Let $\Gamma$ be a graph satisfying the pigeonhole principle but is neither complete nor null.

**Claim 1.** *$\Gamma$ has no isolated vertices.*

*Subproof.* If $\Gamma$ had an isolated vertex, we can partition $\Gamma$ into the set of isolated and the set of non-isolated vertices. But then neither of those two sets is isomorphic to $\Gamma$. ■

By passing to the complement, we also obtain that there are no vertices connected to every other vertex.

Suppose that $\Gamma$ is not isomorphic to $R$. Let $U, V$ be sets such that the witness property fails for $U, V$ with $\#U + \#V$ minimal.

Clearly $m + n > 1$. Partition $U = U_1 \sqcup U_2$, $V = V_1 \sqcup V_2$ such that $U_1 \cup V_1 \neq \varnothing$ and $U_2 \cup V_2 \neq \varnothing$. Let $A := U_1 \cup V_1$ and $B := U_2 \cup V_2$ Let $X$ consist of $A$ together with all vertices outside $B$ which not correctly joined with $U_1, V_1$ and $Y$ consist of $B$ together with all vertices outside $A$ which are not correctly joined with $U_2, V_2$.

By minimality of $\#U + \#V$, we have that neither $X$ nor $Y$ are isomorphic to $\Gamma$. But then $\Gamma \backslash (X \cup Y) \neq \varnothing$ and all $z \in \Gamma \backslash (X \cup Y)$ are correctly joined to $U, V$ $\lightning$. ☐

**Definition 6.3.** A **spanning subgraph** of a graph $G$ is a graph $H$ such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

**Proposition 6.4.** A countable graph $\Gamma$ is isomorphic to a spanning subgraph of $R$ iff for any finite set $\{v_1, \ldots, v_n\} \subseteq I$ there is a vertex $z$ that is adjacent to none of the $v_i$.

**Definition 6.5.** We say that a graph $G$ is **locally finite** iff the neighbourhood of every vertex is finite.

**Proposition 6.6.** Iff the edges of a locally finite graph are deleted from $R$, the result is still isomorphic to $R$.

**Proposition 6.7.** The set $E(R)$ can be partitioned into spanning subgraphs isomorphic to any given countable list of locally finite graphs.

Let $\sigma_{m,n}$ be the sentences saying that a graph has the witness property for $\#U = m, \#V = n$, i.e.

$$\forall u_1, \ldots, u_m, v_1, \ldots, v_n. \\ \left(\bigwedge u_i \neq v_j \wedge \bigwedge u_i \neq u_j \wedge \bigwedge v_i \neq v_j\right) \to \left(\exists z. \bigwedge z \sim u_i \wedge \bigwedge z \nsim v_i\right).$$

**Definition 6.8** (Erdős-Rényi-Gilbert Model)**.** Let $G(n, p)$ denote a random random variable that graph with $n$ vertices and edges added independently with probability $p$.

**Theorem 6.9** (Zero-One Law)**.** Let $\theta$ be a first-order sentence. In the language of graph theory the following are equivalent:

(a) $\theta$ holds in almost all finite random graphs, i.e. $\lim_{N\to\infty} \mathbb{P}[G(N,p) \models \theta] = 1$ for $p \in (0,1)$ fixed.

(b) $R \models \theta$.

(c) $\{\sigma_{m,n} | m, n \in \mathbb{N}\} \models \theta$.

*Proof.* (b) $\iff$ (c) follows from the compactness theorem and the fact that the $\sigma_{m,n}$ axiomatize $R$. (c) $\implies$ (a) We have

$$\mathbb{P}[G(N,p) \models \sigma_{n,m}] \geqslant 1 - N^{m+n}(1 - \frac{1}{2^{m+n}})^{N-m-n} \xrightarrow{N\to\infty} 0.$$

If (c) holds, $\theta$ is a consequence of a finite subset of $\{\sigma_{m,n} | m, n \in \mathbb{N}\}$ since proofs are finite. Hence

$$\mathbb{P}[G(N,p) \models \theta] \xrightarrow{N\to\infty} 1.$$

$(a) \implies (c)$: Suppose $(c)$ fails. Then $\{\sigma_{m,n} | m, n \in \mathbb{N}\} \models \neg\theta$ and we can apply (c) $\implies$ (a) to $\neg\theta$. $\square$

Almost all (in the sense of Baire) countable random graphs have the witness

**Proposition 6.10.** If $M$ is a countable homogeneous relational structure. Then almost all (in the sense of Baire) countable structures younger[a] than $M$ are isomorphic to $M$.

[a] i.e. the age is contained in the age of $M$

Age of a relational structure as tree

## 6.1 Ehrenfeucht-Fraïssé Games

Let $\mathcal{L}$ be a language, $\mathcal{M} = (M, \ldots)$, $\mathcal{N} = (N, \ldots)$ be $\mathcal{L}$-structures and $M \cap N = \varnothing$.

**Definition 6.11.** Let $A \subseteq M$, $B \subseteq N$, $f \colon A \to B$. We call $f$ a **partial embedding**, iff $f \cup \{c^{\mathcal{M}}, c^{\mathcal{N}}) : c \text{ constant of } \mathcal{L}\}$ is a bijection preserving all relations and function symbols of $\mathcal{L}$.

**Definition 6.12.** The infinite **Ehrenfeucht-Fraïssé Game** $\mathrm{EHR}_\omega(\mathcal{M}, \mathcal{N})$ is an infinite two player game.

In each round player 1 ("spoiler") chooses an element of $M$ or $N$. Then player 2 ("duplicator") must select an element of the other structure.

Let $m_i \in M$, $n_i \in N$ be the elements chosen in the $i$-th round.

Spoiler wins if $f \colon m_i \mapsto n_i$ is a partial embedding.

A **strategy** for duplicator is a function $\tau\colon (M \cup N)^{<\omega} \to M \cup N$. In the first round if spoiler plays $c_1$, then duplicator plays $\tau(c_1)$. In the second round spoiler play $c_2$ and duplicator responds by playing $\tau(c_1, c_2)$.

$\tau$ is a winning strategy if by following this strategy duplicator always wins.

**Proposition 6.13.** If $\mathcal{M}$ and $\mathcal{N}$ are countable, then the duplicator has a winning strategy iff $\mathcal{M} \cong \mathcal{N}$.

**Definition 6.14.** Let $\mathrm{EHR}_n(\mathcal{M}, \mathcal{N})$ denote the **Ehrenfeucht-Fraïssé Game** with $n$ rounds. This game ends after $n$-rounds and duplicator wins iff $f := \{(m_i, n_i) | i \leqslant n\}$ is a partial embedding.

**Theorem 6.15.** Let $\mathcal{L}$ be a finite relational language and $\mathcal{M}, \mathcal{N}$ $\mathcal{L}$-structures. Then $\mathcal{M} \equiv \mathcal{N}$ iff duplicator has a winning strategy in $\mathrm{EHR}_n(\mathcal{M}, \mathcal{N})$ for all $n \in \mathbb{N}$.

**Lemma 6.16.** One of the players has a winning strategy in $\mathrm{EHR}_n(\mathcal{M}, \mathcal{N})$.

**Definition 6.17.** The **quantifier depth** of a formula $\varphi$ is defined by

- $\mathrm{depth}(\varphi) = 0$ iff $\varphi$ is quantifier free.
- $\mathrm{depth}(\neg\varphi) = \mathrm{depth}(\varphi)$,
- $\mathrm{depth}(\varphi \wedge \psi) = \mathrm{depth}(\varphi \vee \psi) = \max(\mathrm{depth}(\varphi), \mathrm{depth}(\psi))$.
- $\mathrm{depth}(\alpha x\varphi) = \mathrm{depth}(\exists x\varphi) = 1 + \mathrm{depth}(\varphi)$.

**Lemma 6.18.** Let $\mathcal{L}$ be a finite relational language. Then there exists a finite list of $\mathcal{L}$-formulas $\varphi_1, \ldots, \varphi_k$ of depth at most $n$ in free variables $x_1, \ldots, x_l$ such that every formula of depth at most $n$ in free variables $x_1, \ldots, x_l$ is equivalent to some $\varphi_i$.

*Proof.* We use induction on $n$. Let $\tau_i, i \in I$ be the finite set of atomic formula. Every quantifier-free formula $\varphi$ can be written as a boolean combination of the $\tau_i$, hence

$$\varphi \leftrightarrow \bigvee_{X \in S} \left( \bigwedge_{i \in X} \tau_i \wedge \bigwedge_{i \notin X} \neg\tau_i \right)$$

for some $S \subseteq \mathcal{P}(I)$.

For the inductive step, note that every formula of quantifier depth $n+1$ is a boolean combination of formulae of the form $\exists x\varphi$ and $\forall x\varphi$ where $\mathrm{depth}(\varphi) \leqslant n$. □

**Theorem 6.19.** Let $\mathcal{L}$ be a finite relational language, $\mathcal{M}$, $\mathcal{N}$ $\mathcal{L}$-structures. Then duplicator has a winning strategy in $\mathrm{EHR}_m(\mathcal{M},\mathcal{N})$ iff $\mathcal{M} \equiv_n \mathcal{N}$, i.e. $\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi$ for all $\varphi$ with $\mathrm{depth}(\varphi) \leqslant n$.

# 7 Shelah-Spencer I

7th talk, Josia Pietsch, 2024-06-04

*Proof of Theorem 6.19.* First assume that $\mathcal{M} \not\equiv_n \mathcal{N}$. Using induction on $n$, we show that this translates to a winning strategy for spoiler. For $n = 0$, duplicator immediately loses. Now let $n > 0$. Wlog. $\mathcal{M} \models \forall x.\ \varphi(x)$ and $\mathcal{N} \models \exists x. \neg\varphi(x)$ for some $\varphi$ of quantifier depth $n-1$. Then spoiler chooses $v_0 \in \mathcal{N}$ such that $\neg\varphi(v_0)$ holds. Duplicators response $u_0 \in \mathcal{M}$ satisfies $\varphi(u_0)$, so $(\mathcal{N}, v_0) \equiv_{n-1} (\mathcal{M}, u_0)$, and by induction, spoiler has a winning strategy for $\mathrm{EHR}_{n-1}((\mathcal{N}, v_0), \mathcal{M}, u_0))$.

On the other hand suppose $\mathcal{M} \equiv_n \mathcal{N}$. Wlog. spoiler plays $u_0 \in \mathcal{M}$. Let $\varphi_0(x), \ldots, \varphi_m(x)$ be a list of all formulas of quantifier depth less than $n$ up to equivalence as in Lemma 6.18. Let

$$\Phi(x) :\!= \bigwedge_{\substack{i \\ \mathcal{M} \models \varphi_i(u_0)}} \varphi_i(x) \wedge \bigwedge_{\substack{i \\ \mathcal{M} \not\models \varphi_i(u_0)}} \neg\varphi_i(x)\neg\varphi_i(x).$$

Since $\mathcal{M} \models \Phi(u_0)$ and $\mathrm{depth}(\exists x.\ \Phi(x)) = n$, duplicator can play $v_0 \in \mathcal{N}$ such that $\mathcal{N} \models \Phi(v_0)$. We obtain $(\mathcal{M}, u_0) \equiv_{n-1} (\mathcal{N}, v_0)$ and by induction, duplicator has a winning strategy for $\mathrm{EHR}_{n-1}((\mathcal{M}, u_0), (\mathcal{N}, v_0))$. $\square$

Note that if $\mathcal{L}$ is finite and relational, by Lemma 6.18, there exists only finitely many equivalence classes of $\mathcal{L}$-structures under $\equiv_k$. These equivalence classes are called **$k$-Ehrenfeucht-values** and the set of these is denoted $\mathrm{EHRV}_k$.

**Corollary 7.1.** Let $P$ be a property of $\mathcal{L}$-structures. Suppose that for all $k$ there exist $\mathcal{L}$-structures $\mathcal{M} \equiv_k \mathcal{N}$ such that $\mathcal{M}$ has $P$ but $\mathcal{N}$ does not have $P$. Then $P$ is not first order expressible.

*Proof.* If $P$ is first order expressible, it is expressible by a formula of finite quantifier depth. $\square$

**Theorem 7.2** ([Spe01, p. 3.3.3])**.** For $s \geqslant 3$, $s$-colorability is not first order expressible.

*Proof.* Let $G$ be a connected graph of chromatic number $> s$ and girth $> 3^{k+1}$. Such a graph exists by a theorem of Erdős.

Let $\mathcal{M}$ be the universal cover of $G$ and let $\mathcal{N}$ be $k$-many copies of $G$. Clearly $\mathcal{M}$ has chromatic number 2.

We show that $\mathcal{M} \equiv_k \mathcal{N}$. Let us show by induction, that Duplicator can ensure that

- $u_i$ is a lift of $v_i$ and
- after $n$ moves, the $d_n := \frac{3^{k-n}-1}{2}$ neighborhoods of $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are isomorphic, i.e. there exists an isomorphism of graphs

  $$f \colon \{x \in V(\mathcal{M}) | \min_{1 \leqslant i \leqslant n} \mathrm{dist}(u_i, x) \leqslant d_n\} \leftrightarrow \{y \in \mathcal{N} | \min_{1 \leqslant i \leqslant n} \mathrm{dist}(v_i, y) \leqslant d_n\}$$

  such that $f(u_i) = v_i$.

For the first step this is trivial. Suppose that $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ have been chosen.

- First case: Spoiler chooses $u_{n+1} \in \mathcal{M}$. Let $u_i$ be such that $d := \mathrm{dist}(u_i, u_{n+1})$ is minimal. If $d \leqslant 2 \cdot d_{n+1} + 1$, let $v_{n+1}$ be the image of $u_{n+1}$ in the connected component of $v_n$.

  Otherwise let $v_{n+1}$ be the image of $u_{n+1}$ in a fresh component of $\mathcal{N}$.

- Second case: Spoiler chooses $v_{n+1} \in \mathcal{N}$. If $\mathrm{dist}(v_{n+1}, v_i) > 2 \cdot d_{n+1} + 1$ for all $i \leqslant n$, let $u_{n+1}$ be a lift of $v_{n+1}$ with $\mathrm{dist}(u_i, u_n) > 3^k$ for all $i \leqslant n$. Otherwise let $v_i$ be of minimal distance to $v_{n+1}$ and let $u_{n+1}$ be the endpoint of the lift of the shortest path from $v_i$ to $v_{n+1}$.

□

## 7.1 Random Graphs

**Definition 7.3.** Let $G(n, p)$ denote a random variable, that is a graph with $n$ vertices, where edges are added independently with probability $p$.

Recall:

**Theorem 7.4** (Fagin-GKLT, [Spe01, p. 0.1.2])**.** Let $p \in (0, 1)$.

$$\lim_{n \to \infty} \mathbb{P}[G(n, p) \models A] \in \{0, 1\}$$

for all first order sentences $A$.

**Question 7.4.11.** What happens when $p$ depends on $n$?

We say that some event $A$ happens **almost surely** iff $\lim_{n \to \infty} \mathbb{P}[G(n, p(n)) \models A] = 1$. Given $p(n)$, the **almost sure theory** $\mathcal{T}_p$ is the set of first order sentences $A$ such that $\lim_{n \to \infty} \mathbb{P}[G(n, p(n)) \models A] = 1$. Clearly it is consistent. A function $f$ is called a **threshold function** for $A$ iff $p \ll f \implies A \in \mathcal{T}_p$ and $f \ll p \implies A \notin \mathcal{T}_p$.

We say that $p(n)$ **satisfies the Zero-One Law** iff $\lim_{n\to\infty} \mathbb{P}[G(n,p(n)) \models A] \in \{0,1\}$ for all first order sentences $A$, i.e. iff the almost sure theory is a complete theory.

**Notation 7.4.12.** Let $f, g\colon \mathbb{N} \to \mathbb{R}$. We write

- $f = O(g)$ for $\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$,
- $f = o(g)$ or $f \ll g$ for $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
- $f \sim g$ for $\lim \frac{f(n)}{g(n)} = 1$,
- $f = \Theta(g)$ for $f = O(g)$ and $g = O(f)$.

### 7.1.1 The Erdős-Rényi-Evolution

Let $H$ be a graph with $v$ vertices and $e$ edges. Let $X_H$ denote the number of copies of $H$ in $G(n,p)$.[6] Clearly $\mathbb{E}[X_H] = \Theta(n^v p^e)$. So for $p \ll n^{-v/e}$, $H$ does almost surely not appear. However from $\lim_{n\to\infty} \mathbb{E}[X_H] = \infty$ it does not follow that $H$ is a subgraph almost surely. Indeed if $H$ has a subgraph $H'$ with $v'$ vertices and $e'$ edges, such that $\frac{v'}{e'} < \frac{v}{e}$, the $H$ does not appear for $p \ll n^{-v'/e'}$.

A technique to show that $H$ appears as a subgraph almost surely, is the so-called **second moment method**, i.e. showing $\frac{\mathrm{Var}(X_H)}{\mathbb{E}[X_H]^2} = o(1)$ and applying Chebyshev's Inequality.

**Definition 7.5.** $H$ is **balanced** iff

$$\frac{\#E(H)}{\#V(H)} = \max_{\varnothing \neq H' \subseteq H} \frac{\#E(H')}{\#V(H)}.$$

Erdős and Rényi showed that the threshold function for a *balanced* graph $H$ to occur is $n^{-v/e}$.

**Theorem 7.6** (Zero-One Laws for Very Sparse Graphs.[SS88, Thm. 1])**.**

[6] as a subgraph, not as an induced subgraph

If either

$$\begin{aligned}
&\text{(a)} && p \ll \frac{1}{n^2},\\
&\text{(b)} \quad \exists k \in \mathbb{N}.\ && n^{-\frac{k}{k-1}} \ll p \ll n^{-\frac{k+1}{k}},\\
&\text{(c)} \quad \forall \varepsilon > 0.\ && \frac{1}{n^{1+\varepsilon}} \ll p \ll \frac{1}{n},\\
&\text{(d)} && \frac{1}{n} \ll p \ll \frac{\log n}{n},\\
&\text{(e)} \quad \forall \varepsilon > 0.\ && \frac{\log n}{n} \ll p \ll \frac{1}{n^{1-\varepsilon}},
\end{aligned}$$

then $p$ satisfies the Zero-One law.

*Proof (sketch).* Calculating the expected numbers of the respective structures and using the second moment method, the almost sure theories can be axiomatized as follows:

(a) There are no edges.

(b) All trees on $\leqslant k$ nodes occur as components arbitrarily often, but neither larger connected components nor cycles exist.

(c) All finite trees occur as components arbitrarily often (there might be infinite trees as well, but first order sentences can't capture this.) There are no cycles.

(d) All finite trees occur as components arbitrarily often. All cycles occur as subgraphs arbitrarily often. Every connected component contains at most one cycle. In a connected component containing a cycle, every vertex has infinite degree.

(e) All vertices have infinite degree. Every connected component contains at most one cycle and cycles of all sizes can be found as subgraphs arbitrarily often.

Using Ehrenfeucht games it can be shown that these almost sure theories are indeed complete. □

### 7.1.2 The Main Theorem

**Observe.** *The threshold functions for a graph $H$ to occur is of the form $n^{-\frac{v}{e}}$ (cf. subsubsection 7.1.1). In particular, the exponent is rational.*

**Theorem 7.7** ([Spe01, Thm 1.4.1])**.** Let $\alpha \in (0,1)\backslash\mathbb{Q}$. Then $p(n) := n^{-\alpha}$ satisfies the Zero-One Law, i.e.

$$\lim_{n\to\infty} \mathbb{P}[G(n, n^{-\alpha}) \models A] \in \{0,1\}$$

for all first order sentences $A$.

For the rest of the talk, fix some $\alpha \in (0,1)\backslash\mathbb{Q}$.

## 7.2 Preparations for the Proof of the Zero-One-Law for Irrational Exponents (7.7)

### 7.2.1 Connection to Ehrenfeucht Games

**Theorem 7.8** (Bridge Theorem, [Spe01, Thm 2.5.1])**.** $p$ satisfies the Zero-One-Law iff

$$\lim_{m,n\to\infty} \mathbb{P}[\text{Duplicator wins } \mathrm{EHR}_k(G(n,p(n)),G(m,p(m)))] = 1$$

for all $k \in \mathbb{N}$.

*Proof.* Assume $p$ does not satisfy the Zero-One Law, i.e. $\lim_{n\to\infty} \mathbb{P}[G(n,p(n)) \models A] \notin \{0,1\}$ for some $A$. Then there exists $\varepsilon > 0$ such that

$$\forall N \exists n,m > N.\ \mathbb{P}[G(n,p(n)) \models A] > \varepsilon \wedge \mathbb{P}[G(m,p(n)) \models A] \leqslant 1-\varepsilon.$$

From Theorem 6.19 it follows that Spoiler wins $\mathrm{EHR}_k(G(n,p(n)),G(m,p(m)))$ with probability at least $\varepsilon^2$.

For the converse, assume that $p$ satisfies the Zero-One Law. Fix some $k$. Let $\alpha \in \mathrm{EHRV}_k$ and let $A_\alpha$ be sentence such that $\mathcal{M} \models A_\alpha$ iff $\mathcal{M} \in \alpha$ (this exists by Lemma 6.18). Set $\varepsilon_\alpha := \lim_{n\to\infty} \mathbb{P}[G(n,p(n) \models A_\alpha] \in \{0,1\}$. Since $\mathrm{EHRV}_k$ is finite, we get

$$\sum_\alpha \varepsilon_\alpha = \sum_\alpha \lim_{n\to\infty} \mathbb{P}[G(n,p(n)) \models A_\alpha] = \lim_{n\to\infty} \underbrace{\sum_\alpha \mathbb{P}[G(n,p(n)) \models A_\alpha]}_{1} = 1.$$

Hence there exists exactly one $\beta$ such that $\varepsilon_\beta = 1$ and

$$\begin{aligned} &\lim_{m,n\to\infty} \mathbb{P}[G(n,p(n)) \equiv_k G(m,p(n))] \\ \geqslant &\lim_{m,n\to\infty} \mathbb{P}[G(n,p(n)) \models A_\beta \wedge G(m,p(m)) \models A_\beta] \\ = &\ 1. \end{aligned}$$

□

### 7.2.2 Rooted Graphs

We need a more quantitative way of talking about the extension property:

**Definition 7.9.** A **rooted graph** is a pair $(R, H)$, such that $H$ is a graph and $R \subsetneq V(H)$. $R$ is called the set of **roots**. The **type** of $(R, H)$ is defined to be

$$(|V(H)\backslash R|, |\{e \in E(H)|e\backslash R \neq \varnothing\}|).$$

**Definition 7.10.** Let $R = \{a_1, \ldots, a_r\}$,$V(H)\backslash R = \{b_1, \ldots, b_n\}$. Consider a graph $G$. Let $\overline{x} \subseteq V(G)$ be a set of $r$ vertices. We say that a set of $n$ vertices $\overline{y}$ is an **$(R, H)$-extension** iff $r_i \sim_H v_j \implies x_i \sim_G y_j$ and $v_i \sim_H v_j \implies y_i \sim_G y_j$.

The $(R, H)$ extension statement, $\text{Ext}(R, H)$, says that for any $r$ vertices there exists an $(R, H)$-extension, Formally

$$\forall x_1, \ldots, x_r \left[ \bigwedge_{i \neq j} x_i \neq x_j \to \exists y_1, \ldots, y_v. \left[ \bigwedge_{i \neq j} y_i \neq y_j \wedge \bigwedge_{i,j} x_i \neq y_j \wedge \bigwedge_{\{a_i, b_j\} \in H} x_i \sim y_j \wedge \bigwedge_{\{b_j, b_k\} \in H} y_j \sim y_k \right] \right]$$

**Definition 7.11.** Let $(R, H)$ be a rooted graph.

- Let $R \subsetneq S \subseteq V(H)$. Then $(R, H|_S)$ is called a **subextension**.
- Let $R \subseteq S \subsetneq V(H)$. Then $(S, H)$ is called a **nailextension**.

**Definition 7.12.** Let $(R, H)$ be a rooted graph of type $(v, e)$. We call $(R, H)$

- **dense** iff $v - e\alpha < 0$,
- **sparse** iff $v - e\alpha > 0$,
- **rigid** iff all of its nail extensions are dense,
- **safe** iff all of its subextensions are sparse.
- **minimally safe** iff it is save and there exists no $S$ with $R \subsetneq S \subsetneq V(H)$.

Note that of a fixed set of vertices $\overline{x}$, the expected number $(R, H)$-extensions of $\overline{x}$ goes to $\infty$ if $(R, H)$ is sparse and to 0 if $(R, H)$ is dense. However the notions of sparsity and density do not give the full picture (cf. Definition 7.5), hence two additional notions are needed.

**Fact 7.12.13.** 1. If $(H_0, H_1)$ and $(H_1, H_2)$ are dense, then $(H_0, H_2)$ is dense.

2. If $(H_0, H_1)$ and $(H_1, H_2)$ are sparse, then $(H_0, H_2)$ is sparse.
3. Any sparse extension $(R, H)$ has a safe nailextension.

   Indeed take $S \subsetneq V(H)$ maximal such that $(R, S)$ is dense. Then $(S, H)$ is safe by 1.
4. Any dense extension has a rigid subextension.
5. If $(R, H)$ is not rigid, it has a safe nailextension $(R', H)$.

   (Use 3.)
6. If $(R, H)$ is not safe, it has a rigid subextension $(R, H')$.
7. If $(H_0, H_1)$ and $(H_1, H_2)$ are rigid, then $(H_0, H_2)$ is rigid.

   Let $H_0 \subseteq S \subsetneq H_2$. Then $(H_1 \cap S, H_1)$ is dense, hence $(S, S \cup H_1)$ is dense. Since $(H_1, H_2)$ is rigid, we have that $(S \cup H_1, H_2)$ is dense. Thus $(S, H_2)$ is a dense extension of a dense extension, hence dense.
8. If $(H_0, H_1)$ and $(H_1, H_2)$ are safe, then $(H_0, H_2)$ is safe.

**Definition 7.13.** Fix $t \in \mathbb{Z}$. Let $G$ be a finite graph and $U \subseteq V(G)$. The **$t$-closure** of $U$, $\mathrm{cl}_t(U)$, is the minimal $X \subseteq V(G)$ such that

- $U \subseteq X$ and
- there does not exist and $(R, H)$-extension of $X$ of type $(v, e)$ with $v \leqslant t$.

$\mathrm{cl}_t(U)$ is called **trivial** iff the elements of $U$ are nonadjacent and $\mathrm{cl}_t(U) = U$.

$\mathrm{cl}_t(U)$ is a rigid extension of $U$, as it can be written as a tower of rigid extensions.

**Definition 7.14.** We write $\mathrm{cl}_t(\overline{x}) \cong \mathrm{cl}_t(\overline{y})$ iff there exists a graph isomorphism sending $x_i$ to $y_i$. The **$t$-type** of $\overline{x}$ is the equivalence class of $\mathrm{cl}_t(\overline{x})$.

Note that $\mathrm{cl}_t(\overline{x}) \cong H$ can be written as a first-order formula.

**Theorem 7.15.** Fix $t, r \in \mathbb{N}$. Then $cl_t(x_1, \ldots, x_r)$ is trivial almost surely.

*Proof.* Almost surely, the $x_i$ are not adjacent. The expected number of rigid extensions of type $(v, \_)$, $v \leqslant t$ is $o(1)$. □

**Remark 7.15.14.** It is unlikely that *all* $\overline{x}$ have trivial $t$-closure: Let $\alpha = \frac{\pi}{7}$. Consider a vertex $y$. With high probability $y$ has neighbours $x_1, x_2, x_3$, but that $\overline{x}$ has non-trivial 1-closure.

**Theorem 7.16** (Finite Closure Theorem, [Spe01, Thm. 4.3.2])**.** For fixed $t, r \in \mathbb{N}$ there is a constant $K$, such that a.s. $|\operatorname{cl}_t(x_1, \ldots, x_r)| \leqslant r + K$ for

*Proof.* Set

$$\begin{aligned} \beta &:= \min_{\substack{(v,e) \\ v \leqslant t \\ v - e\alpha < 0}} \frac{e\alpha - v}{v} \\ K &:= \left\lceil \frac{r}{\beta} \right\rceil. \end{aligned}$$

Suppose $\# \operatorname{cl}_t(R) \geqslant K$ for some $R$ with $\#R = r$. Then we have

$$R = S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_u \subseteq \operatorname{cl}_t(R)$$

such that

- each $(S_i, S_{i+1})$ is rigid of type $(v_i, e_i)$ and
- $K \leqslant \sum_{i=0}^{u-1} v_i \leqslant K + t$.

Then $S_u$ has $V = r + \sum v_i$ vertices and $E = \sum e_i$ edges. The expected number of such graphs is $O(n^{V - \alpha E})$, but

$$V - \alpha E = r + \sum_i (v_i - \alpha e_i) \leqslant r - \sum_i \beta v_i \leqslant r - K\beta < 0.$$

Since there are only finitely many graphs that might occur as $S_u$, this finishes the proof. $\square$

**Theorem 7.17** ([Spe01, Thm. 4.3.3])**.** Almost surely $\operatorname{cl}_t(\varnothing) = \varnothing$.

*Proof.* Set $r = 0$ in Theorem 7.15. $\square$

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# References


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