Seminar: Model Theory and Combinatorics

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Contents

These are my notes on the Seminar Model Theory and Combinatorics taught by Prof. Martin Hils and Dr. Rob Sullivan in the summer term 2024 at the University Münster.

Warning 0.1. This is not an official script.

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1 Introduction and Ultrahomogeneity

1[st] talk, Rob Sullivan, 2024-04-09

Theorem 1.1 (Erdős-Rényi)**.** Slogan: "Any two countably infinite random graphs are almost surely isomorphic".

Let $0 < p < 1$. Then there exists a countably infinite graph R such that if we produce a graph Γ with $V(\Gamma) := \mathbb{N}$ and for each pair of vertices we put an edge with probability p then $\mathbb{P}[\Gamma \cong R] = 1$.

This graph R is called **the random graph**.

Proof of Theorem 1.1. We say that a graph Γ has the **witness property** (WP) iff for all pairs (U, V) of finite disjoint subsets of $V(\Gamma)$, we have $w \in \Gamma\backslash(U \cap V)$, such that w is adjacent to all elements of U and none of V.

Any graph with the witness property is infinite.

The proof is split into two steps:

Claim 1.1.1. $\mathbb{P}[\Gamma$ *has WP*$] = 1$.

Claim 1.1.2. *Any two countably infinite graphs with the witness property are isomorphic.*

Proof of Claim 1.1.1. We have

$$\begin{aligned}\mathbb{P}[\Gamma \text{ does not have WP}] &= \mathbb{P}[\exists U, V \text{ with no witness}]\\ &\leqslant \sum_{(U,V)} \mathbb{P}[(U,V) \text{ has no witness}]\end{aligned}$$

For a fixed node $w \in V(\Gamma)\backslash(U \cap V)$, the probability that w is not a witness for (U, V) is $1 - p^{|U|}(1-p)^{|V|}$, hence $\mathbb{P}[(U, V)$ have no witness$] = 0$. □

Explicit construction We give an explicit construction of a graph with the WP:[1] Let $\mathbb{N}$ be the set of nodes. For $x, y \in \mathbb{N}$, we set $x \sim y$ iff the xth digit of the binary expansion of y is 1 or the yth digit of the binary expansion of x is 1.[2]

For U, V finite and disjoint, let $n := \max(U \cap V)$ and take $w \geqslant 2^n$ such that the binary digits of w are set appropriately.

Proof of Claim 1.1.2. Now let Γ and Γ' be countable graphs with the WP. We prove by "back-and-forth"[3], that they are isomorphic:

Enumerate $V(\Gamma)$ as $m_0, m_1, \ldots$ and $V(\Gamma')$ as $m_0', m_1', \ldots$ We construct a sequence of partial isomorphisms f_n (i.e. isomorphisms of a finite subset of Γ to a finite subset of Γ') such that

(i) $f_{n+1} \supseteq f_n$,

(ii) $m_t \in \operatorname{dom} f_{2t}$,

(iii) $m_t' \in \operatorname{dom} f_{2t+1}$.

Given such partial isomorphisms, we can take $f := \bigcup f_t$.

Suppose we already have f_{2t-1}. If $m_t \in f_{2t-1}$ let $f_{2t} := f_{2t-1}$. Otherwise let U be the neighbourhood of m_t and $V := (\operatorname{dom} f_{2t-1}) \backslash U$. Choose a witness w for $f_{2t-1}(U), f_{2t-1}(V)$ and set $f_{2t}(m_t) := w$. □

□

The proof of Claim 1.1.2 was very similar to the proof of

Theorem 1.2 (Cantor)**.** Any two countably infinite DLOWEs[a] are isomorphic.

[a]dense linear orders without endpoints

Dense and without endpoints is equivalent to:

For any $A \subseteq_{\text{fin}} M$ and any partition $A = U \cup V$, where $U < V$[4] there exists $w \in M$ with $U < w < V$.

Idea. *We want to generalize this.*

Let $\mathcal{L}$ be a countable relational language and M and $\mathcal{L}$-structure.

Definition 1.3. The age of M is the class of finite structures embeddable

[1]This is not necessary for the proof.
[2]$0 \in \mathbb{N}$ and indices start at 0
[3]maybe it should be called "forth-and-back"
[4]i.e. $\forall u \in U, v \in V.\ u < v$

into M.

We say that M has the **extension property** (**EP**) iff

$$\begin{aligned}
&\forall A \subseteq_{\text{fin}} M. \\
&\ \forall \text{embedding } f\colon A \to B \text{ where } B \in \operatorname{Age}(M). \\
&\ \ \exists \text{embedding } h\colon B \to M. \\
&\ \ \ \forall a \in A.\ hf(a) = a.
\end{aligned}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[r, phantom, "\subseteq"] & M \\
B \arrow[ur, dashed, "\exists h"']
\end{tikzcd}$$

Example 1.4. A simple induction shows that the age of R is the set of all finite graphs. The random graph has the extension property.

The extension property was the key ingredient to the proof of Theorem 1.1.

Proposition 1.5. Let M, M' be countable $\mathcal{L}$-structures with the same age and the EP. Then $M \cong M'$.

Proof. This is same as the proof of Claim 1.1.2. □

Remark 1.5.1. One needs to be more careful if $\mathcal{L}$ contains constants and function symbols.

Proposition 1.6. Let M and M' be countable $\mathcal{L}$-structures with the same age and the EP.

Then any partial isomorphism $A \to A'$ where A and A' are finite substructures of M resp. M' extends to an isomorphism $M \to M'$.

Proof. Use $A \to A'$ as a starting point for the back-and-forth in the proof of Proposition 1.5. □

Definition 1.7. An $\mathcal{L}$-structure M is called **ultrahomogeneous**[a] iff any isomorphism $f\colon A \to A'$ with $A, A' \subseteq_{\text{fin}} M$ extends to an automorphism of M.

[a] sometimes also **homogeneous**, but not in this seminar

Proposition 1.8. Let M be countable. Then M has the EP iff M is ultrahomogeneous.

Proof. "$\implies$" Take $M = M'$ in Proposition 1.6.

"$\impliedby$" Let A, B be substructures of M $f: A \to B$. Then $f: A \to f(A)$ is isomorphism, hence it extends to an isomorphism h of M and we can take

$$\begin{tikzcd} A \arrow[r, phantom, "\subseteq"] \arrow[d, "f"'] & M \\ B \arrow[ur, dashed, "h^{-1}"'] & \end{tikzcd}$$

missing image

$\square$

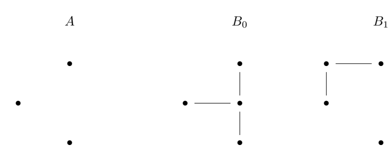
Question 1.8.2. Can we detect ultrahomogeneity from the age?

Question 1.8.3. When do we have these nice universal[a] objects, i.e. what classes of structures are ages of an ultrahomogeneous model?

[a]sometimes called generic

Example 1.9. Let X be the class of finite forests.[a]

Suppose that M is a countable ultrahomogeneous structure with age k.



We have $A \subseteq M$. $A \hookrightarrow B_0$ and $A \hookrightarrow B_1$. But now we get a contradiction, since simultaneously embedding A into a copy B_0 and B_1, we get a circle.

[a]A **forest** is a graph without cycles. A **tree** is a connected forest.

Definition 1.10. A class $\mathcal{K}$ of finite $\mathcal{L}$-structures has the **amalgamation property** (**AP**) iff for all embeddings $f: A \to B_0$, $f_1: A \to B_1$ of finite structures in $\mathcal{K}$, there exists $C \in \mathcal{K}$ with embeddings $h_0: B_0 \to C$, $h_1: B_1 \to C$ such that $\forall a \in A.\ h_0 f_0(a) = h_1 f_1(a)$

$$
\begin{tikzcd}
 & B_0 \arrow[dr, dashed, "h_0"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & C \\
 & B_1 \arrow[ur, dashed, "h_1"'] &
\end{tikzcd}
$$

The AP is the real meat.

Definition 1.11. We say that $\mathcal{K}$ has **JEP** (**joint embedding property**) iff for all $A_0, A_1 \in \mathcal{K}$, there exists $B \in \mathcal{K}$ and embeddings $f_0 \colon A_0 \to B$, $f_1 \colon A_1 \to B$.

$$
\begin{tikzcd}
A_0 \arrow[dr, dashed, "f_0"] & \\
 & B \\
A_1 \arrow[ur, dashed, "f_1"'] &
\end{tikzcd}
$$

The JEP is a property we expect from every reasonable class.

Proposition 1.12. Let M be a countable ultrahomogeneous structure. Then $\operatorname{Age}(M)$ has AP.

Proof. Consider

$$
\begin{tikzcd}
 & B_0 \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & \\
 & B_1
\end{tikzcd}
$$

We can assume $B_0, B_1 \subseteq M$. The isomorphism $f_0(a) \mapsto f_1(a)$ can be extended to $h \in \operatorname{Aut}(M)$. Then

missing picture

$$
\begin{tikzcd}
 & B_0 \arrow[dr, "h"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & h(B_0) \cup B_1 \\
 & B_1 \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}
$$

completes the diagram. $\square$

Definition 1.13. A hereditary[a] class $\mathcal{K}$ of finite $\mathcal{L}$-structures with the AP, JEP and countably many isomorphism types is called an **amalgamation class** (or **Fraïssé class**).

[a] closed under taking substructures, i.e. if $B \in \mathcal{K}$ and $A \hookrightarrow B$ then $A \in \mathcal{K}$

Theorem 1.14 (Fraïssé's Theorem)**.** Let $\mathcal{K}$ be an amalgamation class. Then there exists a countable ultrahomogeneous M with $\mathrm{Age}(M) = \mathcal{K}$.

Definition 1.15. Let $\mathcal{K}$ be an amalgamation class. An increasing chain $F_0 \subseteq F_1 \subseteq \ldots \subseteq$ of structures in $\mathcal{K}$ is called a **Fraïssé sequence** (or **rich sequence**) iff for each F_i an for each $A \subseteq F_i$ and each embedding $f : A \to B$ with $B \in \mathcal{K}$, there exists $j > i$ and an embedding $h : B \to F_j$ such that the diagram commutes.

$$\begin{tikzcd}
A \arrow[r, phantom, "\subseteq"] \arrow[dr, "\forall f"'] & F_i \arrow[r, phantom, "\subseteq"] & F_j \\
& B \arrow[ur, dashed, "\exists h"'] &
\end{tikzcd}$$

Lemma 1.16. Let $\mathcal{K}$ be an amalgamation class. Let $(F_i)_i < \omega$ be a Fraïssé-sequence. Then $M = \bigcup_{i<\omega} F_i$ is ultrahomogeneous and $\mathrm{Age}(M) = \mathcal{K}$.

Proof. Let $A \in \mathcal{K}$. Use the JEP on A and F_0 to get B. Since the F_i form a Fraïssé-sequence, we get $B \hookrightarrow F_k$ for some k.

$$\begin{tikzcd}
F_0 \arrow[dr] & & F_j \\
& B \arrow[ur] & \\
A \arrow[ur] & &
\end{tikzcd}$$

M has the EP:

$$\begin{tikzcd}
A \arrow[r, phantom, "\subseteq"] \arrow[dr] & F_i \arrow[r, phantom, "\subseteq"] & F_j \\
& B \arrow[ur, dashed] &
\end{tikzcd}$$

$\square$

Proof of Theorem 1.14. By Lemma 1.16 it suffices to build a Fraïssé-sequence. Let $\sigma\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection such that if $\sigma(n) = (i, j)$, then $n \geqslant i$. This is called a "scheduling function".

missing image: "set theory"

Take $F_0 \in \mathcal{K}$. List all embeddings $f_{0,0}, f_{0,1}, \dots$ of finite substructures of F_0 into other structures of $\mathcal{K}$ (up to isomorphism).

At stage F_k, deal with the embedding $f_{\sigma(k)}$ using AP.

$$
\begin{tikzcd}
A \arrow[dr, "f_{0,0}"'] & \subseteq & F_0 & \subseteq & F_1 \\
 & & B \arrow[urr] & &
\end{tikzcd}
$$

□

Example 1.17. Consider the class of finite graphs. This has AP: Let $A \hookrightarrow B_0$, $A \hookrightarrow B_1$. Take the free amalgam of B_0 and B_1 along A.

So by Fraïssé's Theorem (1.14) we get the random graph.

Example 1.18. Consider the class of triangle-free finite graphs. Note that the free amalgam does not create triangles, so Fraïssé's Theorem (1.14) can be applied. The resulting graph is called the **Henson graph** H_3 (or "**triangle-free random graph**"[a]).

[a] this is a bad name

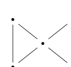
For the Henson graph the analogous statement to the witness property is:

$\forall U, V$ finite and disjoint and U edge free, there exists w such that ...

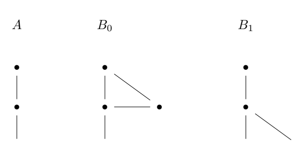
The same thing can also be done the obtain the K_n-free random graph.

Example 1.19. Non-examples are:

- finite forests
- finite bowtie-free graphs, i.e. not containing



as a graph-theoretic subgraph. Consider



Theorem 1.20 (Lachlan-Woodrow)**.** Any ultrahomogeneous countably infinite graph is

- the random graph R,
- H_n, the K_n-free graph or
- the disjoint union of K_ns (where maybe $n = \infty$)

or its complement.

We will not prove this in this seminar.

Index