# Seminar: Model Theory and Combinatorics

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# Contents

These are my notes on the Seminar Model Theory and Combinatorics taught by Prof. Martin Hils and Dr. Rob Sullivan in the summer term 2024 at the University Münster.

**Warning 0.1.** This is not an official script.

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# 1 Introduction and Ultrahomogeneity

1st talk, Rob Sullivan, 2024-04-09

**Theorem 1.1** (Erdős-Rényi)**.** Slogan: "Any two countably infinite random graphs are almost surely isomorphic".

Let $0 < p < 1$. Then there exists a countably infinite graph $R$ such that if we produce a graph $\Gamma$ with $V(\Gamma) := \mathbb{N}$ and for each pair of vertices we put an edge with probability $p$ then $\mathbb{P}[\Gamma \cong R] = 1$.

This graph $R$ is called **the random graph**.

*Proof of Theorem 1.1.* We say that a graph $\Gamma$ has the **witness property** (WP) iff for all pairs $(U, V)$ of finite disjoint subsets of $V(\Gamma)$, we have $w \in \Gamma \setminus (U \cap V)$, such that $w$ is adjacent to all elements of $U$ and none of $V$.

Any graph with the witness property is infinite.

The proof is split into two steps:

**Claim 1.1.1.** $\mathbb{P}[\Gamma \text{ } has \text{ } WP] = 1$.

**Claim 1.1.2.** *Any two countably infinite graphs with the witness property are isomorphic.*

*Proof of Claim 1.1.1.* We have

$$\begin{aligned}\mathbb{P}[\Gamma \text{ does not have WP}] &= \mathbb{P}[\exists U, V \text{ with no witness}]\\ &\leqslant \sum_{(U,V)} \mathbb{P}[(U,V) \text{ has no witness}]\end{aligned}$$

For a fixed node $w \in V(\Gamma) \setminus (U \cap V)$, the probability that $w$ is not a witness for $(U, V)$ is $1 - p^{|U|}(1-p)^{|V|}$, hence $\mathbb{P}[(U, V) \text{ have no witness}] = 0$. □

**Explicit construction** We give an explicit construction of a graph with the WP:[1] Let $\mathbb{N}$ be the set of nodes. For $x, y \in \mathbb{N}$, we set $x \sim y$ iff the $x^{\text{th}}$ digit of the binary expansion of $y$ is 1 or the $y^{\text{th}}$ digit of the binary expansion of $x$ is 1.[2]

For $U, V$ finite and disjoint, let $n := \max(U \cap V)$ and take $w \geqslant 2^n$ such that the binary digits of $w$ are set appropriately.

*Proof of Claim 1.1.2.* Now let $\Gamma$ and $\Gamma'$ be countable graphs with the WP. We prove by "back-and-forth"[3], that they are isomorphic:

Enumerate $V(\Gamma)$ as $m_0, m_1, \ldots$ and $V(\Gamma')$ as $m_0', m_1', \ldots$ We construct a sequence of partial isomorphisms $f_n$ (i.e. isomorphisms of a finite subset of $\Gamma$ to a finite subset of $\Gamma'$) such that

(i) $f_{n+1} \supseteq f_n$,

(ii) $m_t \in \operatorname{dom} f_{2t}$,

(iii) $m_t' \in \operatorname{dom} f_{2t+1}$.

Given such partial isomorphisms, we can take $f := \bigcup f_t$.

Suppose we already have $f_{2t-1}$. If $m_t \in f_{2t-1}$ let $f_{2t} := f_{2t-1}$. Otherwise let $U$ be the neighbourhood of $m_t$ and $V := (\operatorname{dom} f_{2t-1}) \backslash U$. Choose a witness $w$ for $f_{2t-1}(U), f_{2t-1}(V)$ and set $f_{2t}(m_t) := w$. □

□

The proof of Claim 1.1.2 was very similar to the proof of

**Theorem 1.2** (Cantor)**.** Any two countably infinite DLOWEs[a] are isomorphic.

[a]dense linear orders without endpoints

Dense and without endpoints is equivalent to:

For any $A \subseteq_{\text{fin}} M$ and any partition $A = U \cup V$, where $U < V$[4] there exists $w \in M$ with $U < w < V$.

**Idea.** *We want to generalize this.*

Let $\mathcal{L}$ be a countable relational language and $M$ and $\mathcal{L}$-structure.

**Definition 1.3.** The age of $M$ is the class of finite structures embeddable into $M$.

[1]This is not necessary for the proof.
[2]$0 \in \mathbb{N}$ and indices start at 0
[3]maybe it should be called "forth-and-back"
[4]i.e. $\forall u \in U, v \in V.\ u < v$

We say that $M$ has the **extension property** (**EP**) iff

$$\begin{aligned}
&\forall A \subseteq_{\text{fin}} M.\\
&\quad \forall f\colon A \to B \text{ embedding where } B \in \operatorname{Age}(M).\\
&\quad\quad \exists h\colon B \to M \text{ embedding.}\\
&\quad\quad\quad \forall a \in A.\ hf(a) = a
\end{aligned}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[r, phantom, "\subseteq"] & M \\
B \arrow[ur, dashed, "\exists h"']
\end{tikzcd}$$

**Example 1.4.** A simple induction shows that the age of $R$ is the set of all finite graphs. The random graph has the extension property.

The extension property was the key ingredient to the proof of Theorem 1.1.

**Proposition 1.5.** Let $M, M'$ be countable $\mathcal{L}$-structures with the same age and the EP. Then $M \cong M'$.

*Proof.* This is same as the proof of Claim 1.1.2. $\square$

**Remark 1.5.1.** One needs to be more careful if $\mathcal{L}$ contains constants and function symbols.

**Proposition 1.6.** Let $M$ and $M'$ be countable $\mathcal{L}$-structures with the same age and the EP.

Then any partial isomorphism $A \to A'$ where $A$ and $A'$ are finite substructures of $M$ resp. $M'$ extends to an isomorphism $M \to M'$.

*Proof.* Use $A \to A'$ as a starting point for the back-and-forth in the proof of Proposition 1.5. $\square$

**Definition 1.7.** An $\mathcal{L}$-structure $M$ is called **ultrahomogeneous**[a] iff any isomorphism $f\colon A \to A'$ with $A, A' \subseteq_{\text{fin}} M$ extends to an automorphism of $M$.

[a] sometimes also **homogeneous**, but not in this seminar

**Proposition 1.8.** Let $M$ be countable. Then $M$ has the EP iff $M$ is ultrahomogeneous.

*Proof.* " $\Longrightarrow$ " Take $M = M'$ in Proposition 1.6.

" $\Longleftarrow$ " Let $A$, $B$ be substructures of $M$ $f\colon A \to B$. Then $f\colon A \to f(A)$ is isomorphism, hence it extends to an isomorphism $h$ of $M$ and we can take

$$\begin{tikzcd} A \arrow[d, "f"'] & \subseteq & M \\ B \arrow[urr, dashed, "h^{-1}"'] & & \end{tikzcd}$$

missing image

$\square$

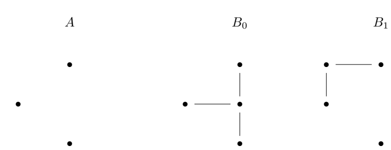
**Question 1.8.2.** Can we detect ultrahomogeneity from the age?

**Question 1.8.3.** When do we have these nice universal[a] objects, i.e. what classes of structures are ages of an ultrahomogeneous model?

[a]sometimes called generic

**Example 1.9.** Let $X$ be the class of finite forests.[a]

Suppose that $M$ is a countable ultrahomogeneous structure with age $k$.




We have $A \subseteq M$. $A \hookrightarrow B_0$ and $A \hookrightarrow B_1$. But now we get a contradiction, since simultaneously embedding $A$ into a copy $B_0$ and $B_1$, we get a circle.

[a]A **forest** is a graph without cycles. A **tree** is a connected forest.

**Definition 1.10.** A class $\mathcal{K}$ of finite $\mathcal{L}$-structures has the **amalgamation property** (**AP**) iff for all embeddings $f\colon A \to B_0$, $f_1\colon A \to B_1$ of finite structures in $\mathcal{K}$, there exists $C \in \mathcal{K}$ with embeddings $h_0\colon B_0 \to C$, $h_1\colon B_1 \to C$ such that $\forall a \in A.\ h_0 f_0(a) = h_1 f_1(a)$

$$\begin{tikzcd}
& B_0 \arrow[dr, dashed, "h_0"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & C \\
& B_1 \arrow[ur, dashed, "h_1"'] &
\end{tikzcd}$$

The AP is the real meat.

**Definition 1.11.** We say that $\mathcal{K}$ has **JEP** (**joint embedding property**) iff for all $A_0, A_1 \in \mathcal{K}$, there exists $B \in \mathcal{K}$ and embeddings $f_0 \colon A_0 \to B$, $f_1 \colon A_1 \to B$.

$$\begin{tikzcd}
A_0 \arrow[dr, dashed, "f_0"] & \\
& B \\
A_1 \arrow[ur, dashed, "f_1"'] &
\end{tikzcd}$$

The JEP is a property we expect from every reasonable class.

**Proposition 1.12.** Let $M$ be a countable ultrahomogeneous structure. Then $\mathrm{Age}(M)$ has AP.

*Proof.* Consider

$$\begin{tikzcd}
& B_0 \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & \\
& B_1
\end{tikzcd}$$

We can assume $B_0, B_1 \subseteq M$. The isomorphism $f_0(a) \mapsto f_1(a)$ can be extended to $h \in \mathrm{Aut}(M)$. Then

missing picture

$$\begin{tikzcd}
& B_0 \arrow[dr, "h"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "f_1"'] & & h(B_0) \cup B_1 \\
& B_1 \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

completes the diagram. □

**Definition 1.13.** A hereditary[a] class $\mathcal{K}$ of finite $\mathcal{L}$-structures with the AP, JEP and countably many isomorphism types is called an **amalgamation class** (or **Fraïssé class**).

[a]closed under taking substructures, i.e. if $B \in \mathcal{K}$ and $A \hookrightarrow B$ then $A \in \mathcal{K}$

**Theorem 1.14** (Fraïssé's Theorem)**.** Let $\mathcal{K}$ be an amalgamation class. Then there exists a countable ultrahomogeneous $M$ with $\mathrm{Age}(M) = \mathcal{K}$.

**Definition 1.15.** Let $\mathcal{K}$ be an amalgamation class. An increasing chain $F_0 \subseteq F_1 \subseteq \ldots \subseteq$ of structures in $\mathcal{K}$ is called a **Fraïssé sequence** (or **rich sequence**) iff for each $F_i$ and for each $A \subseteq F_i$ and each embedding $f\colon A \to B$ with $B \in \mathcal{K}$, there exists $j > i$ and an embedding $h\colon B \to F_j$ such that the diagram commutes.

$$\begin{tikzcd}
A \arrow[drr, "\forall f"'] & \subseteq & F_i & \subseteq & F_j \\
& & B \arrow[urr, dashed, "\exists h"'] & &
\end{tikzcd}$$

**Lemma 1.16.** Let $\mathcal{K}$ be an amalgamation class. Let $(F_i)_i < \omega$ be a Fraïssé-sequence. Then $M = \bigcup_{i<\omega} F_i$ is ultrahomogeneous and $\mathrm{Age}(M) = \mathcal{K}$.

*Proof.* Let $A \in \mathcal{K}$. Use the JEP on $A$ and $F_0$ to get $B$. Since the $F_i$ form a Fraïssé-sequence, we get $B \hookrightarrow F_k$ for some $k$.

$$\begin{tikzcd}
F_0 \arrow[dr] & & F_j \\
& B \arrow[ur] & \\
A \arrow[ur] & &
\end{tikzcd}$$

$M$ has the EP:

$$\begin{tikzcd}
A \arrow[drr] & \subseteq & F_i & \subseteq & F_j \\
& & B \arrow[urr, dashed] & &
\end{tikzcd}$$

□

*Proof of Theorem 1.14.* By Lemma 1.16 it suffices to build a Fraïssé-sequence. Let $\sigma\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection such that if $\sigma(n) = (i, j)$, then $n \geqslant i$. This is called a "scheduling function".

Take $F_0 \in \mathcal{K}$. List all embeddings $f_{0,0}, f_{0,1}, \ldots$ of finite substructures of $F_0$ into other structures of $\mathcal{K}$ (up to isomorphism).

At stage $F_k$, deal with the embedding $f_{\sigma(k)}$ using AP.

$$\begin{tikzcd}
A \arrow[r, phantom, "\subseteq"] \arrow[dr, "f_{0,0}"'] & F_0 \arrow[r, phantom, "\subseteq"] & F_1 \\
& B \arrow[ur] &
\end{tikzcd}$$

$\square$

**Example 1.17.** Consider the class of finite graphs. This has AP: Let $A \hookrightarrow B_0$, $A \hookrightarrow B_1$. Take the free amalgam of $B_0$ and $B_1$ along $A$.

So by Fraïssé's Theorem (1.14) we get the random graph.

**Example 1.18.** Consider the class of triangle-free finite graphs. Note that the free amalgam does not create triangles, so Fraïssé's Theorem (1.14) can be applied. The resulting graph is called the **Henson graph** $H_3$ (or "**triangle-free random graph**"[a]).

[a]this is a bad name

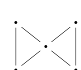
For the Henson graph the analogous statement to the witness property is:

$\forall U, V$ finite and disjoint and $U$ edge free, there exists $w$ such that ...

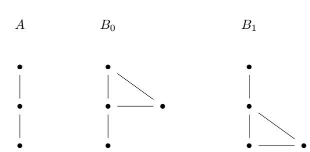
The same thing can also be done the obtain the $K_n$-free random graph.

**Example 1.19.** Non-examples are:

- finite forests
- finite bowtie-free graphs, i.e. not containing



  as a graph-theoretic subgraph. Consider



**Theorem 1.20** (Lachlan-Woodrow)**.** Any ultrahomogeneous countably infinite graph is

- the random graph $R$,
- $H_n$, the $K_n$-free graph or
- the disjoint union of $K_n$s (where maybe $n = \infty$)

or its complement.

We will not prove this in this seminar.

## 2 More Fraïssé: algebraic structures, other examples

2<sup></sup>

2nd talk, 2024-04-16, Zahra Mohammadi

TODO

## 3 $\omega$-Categoricity and more Model Theory

3rd talk, 2024-04-30, Rob Sullivan

We again consider relational structures.

**Definition 3.1.** A Fraïsse structure $M$ has the **strong EP** if

$$
\begin{array}{l}
\forall A \overset{\text{finite}}{\subseteq} M. \\
\quad \forall A \xrightarrow{f} B \text{ with } B \in \operatorname{Age}(M). \\
\quad\quad \forall V \overset{\text{finite}}{\subseteq} M \text{ disjoint from A.} \\
\quad\quad\quad \exists \text{ embedding } h\colon B \to M \\
\quad\quad\quad\quad \text{such that}
\end{array}
$$

$$
\begin{tikzcd}
A \arrow[rr, phantom, "\subseteq"] \arrow[dr, "f"'] & & M \\
& B \arrow[ur, "h"'] &
\end{tikzcd}
$$

and $h$ avoids $V$ (i.e. $\operatorname{im} h \cap V = \varnothing$).

**Lemma 3.2.** Let $M$ be a Fraïsse structure. Then $M$ has the strong EP iff $\operatorname{Age}(M)$ has the strong AP.

*Proof.* □

TODO

**Lemma 3.3.** Let $M$ be a Fraïsse structure. Then $\mathrm{Age}(M)$ has the strong AP iff $\forall V \overset{\text{finite}}{\subseteq} M.\ M \backslash V \cong M$.

*Proof.* $\Longrightarrow$ Take $A \in \mathrm{Age}(M)$. Use the strong EP over $\varnothing$ avoiding $V$ to get a copy of $A$ inside $M \backslash V$. So $\mathrm{Age}(M \backslash V) = \mathrm{Age}(M)$. The strong EP for $M$ implies that $M \backslash V$ has EP. Use Fraïssé's theorem.

$\Longleftarrow$ The condition immediately implies strong EP. □

**Example 3.4.** Let $M$ be the disjoint union of two copies of the random graph. For all $V \overset{\text{finite}}{\subseteq} M$, we have $M \backslash V \cong M$. However $M$ is not ultrahomogeneous.

**Lemma 3.5** (Neumann)**.** Let $G \curvearrowright \Omega$ be a group action without finite orbits.

Then for all $A, B \overset{\text{finite}}{\subseteq} \Omega$, there exists $g \in G$ such that $gA \cap B = \varnothing$.

*Proof.* If $|A| = 1$, this is trivial.

Since there are no finite orbits, for all $A \overset{\text{finite}}{\subseteq} \Omega$ the set $\{gA : g \in G\}$ is infinite.

Fix $B$ and do induction on $|A|$.

We'll only look at some finite cases. Let $|A| = 2$. Suppose that every translate of $A$ intersects $B$. For $x \in \Omega$, let $T_x$ be the set of translates of $A$ containing $x$.

Let $b_0 \in B$. It suffices to show that $T_{b_0}$ is finite (since then it follows that the set of translate of $A$ is finite, which is a contradiction).

By the case of $|A| = 1$, there there exists $g_0 \in G$ such that $g_0 b_0 \notin B$. For $gA \in T_{g_0 b_0}$, we have $gA \cap b \neq \varnothing$, so as $|A| = 2$ and $B$ is finite, so $T_{g_0 b_0}$ is finite, hence $T_{b_0}$ is finite.

Let $|A| = 3$ Let $b_0 \in B$, Then there exists $b_0 \in G$ such that $g_0 b_0 \notin B$. Each $gA \in T_{g_0 b_0}$ has $gA \cap B \neq \varnothing$. So $T_{g_0 b_0} = \bigcup_{b_1 \in B}, T_{g_0 b_0, b_1}$. From the case of $|A| = 2$ , there exists $g_1$ with $g_1 g_0 b_0, g_1 b_1 \notin B$. It follows that $T_{g_1 g_0 b_0, g_1 b_1}$ is finite.

The general case is left as an exercise. □

**Definition 3.6.** Let $G \curvearrowright \Omega$ be a group action and $A \overset{\text{finite}}{\subseteq} \Omega$. The **finite-orbit closure**, $\mathrm{fcl}(A)$, is the union of the finite orbits of the pointwise stabilizers $G_{(A)}$ of $A$.

Note that $A \subseteq \mathrm{fcl}(A)$. We say that $\mathrm{fcl}(A)$ is trivial iff $\mathrm{fcl}(A) = A$.

**Proposition 3.7.** Let $M$ be a Fraïssé-structure. Then $M$ has strong amalgamation iff $\forall A \overset{\text{finite}}{\subseteq} M, \mathrm{fcl}(A) = A$.

*Proof.* " $\implies$ " Take $m \in M \backslash A$. Let $k \geqslant 1$. Take the strong amalgam of $k$ instances of $A \hookrightarrow A \cup \{m\}$. Then use the EP over $A$. So $\mathrm{Orb}_{G_{(A)}}(M)$ is infinite.

" $\impliedby$ "

Work inside $M$. We can assume $A \subseteq B_0 \cap B_1 \subseteq M$. Consider $G_{(A)} \curvearrowright M\backslash A$ and apply Neumann's Lemma (3.5). $\square$

Let's get formulae involved.

**Question 3.7.4.** Recall $\mathcal{L}_{\text{graph}} = \{\sim\}$. What graph-theoretic properties are axiomatizable in first-order logic? I.e. given a graph property $P$, is there a theory $T$ such that $\Gamma \models T \iff \Gamma$ has property $P$ for all graphs $\Gamma$.

- Being bipartite is axiomatizable, since a graph is bipartite iff it does not contain a circle of odd diameter.

  (Note that we can not directly talk about partitions in first-order formulae.)

- Not being bipartite is not aximoatizable, since being bipartite but not containing a cycle of length $\leqslant n$ is consistent.

- Connectedness and not-connectedness are not axiomatizable:

  Let $T$ be the theory saying that every vertex has degree 2 and the graph is acyclic. $T$ has no finite models. If $\Gamma \models T$, then $\Gamma$ consists of disjoint copies of $\mathbb{Z}$. So $T$ is $\kappa$-categorical for $\kappa$ uncountable. Hence $T$ is complete by Vaught's Test (3.9). In particular, $\mathrm{Th}(\mathbb{Z}) = \mathrm{Th}(\mathbb{Z} \sqcup \mathbb{Z})$.

**Definition 3.8.** Let $\mathcal{L}$ be a language (not necessarily relational). Let $\kappa \geqslant |\mathcal{L}|$ be a cardinal. Let $T$ be an $\mathcal{L}$-theory with infinite models. Then $T$ is **$\kappa$-categorical** iff all models of $T$ of cardinality $\kappa$ are isomorphic.

**Theorem 3.9** (Vaught's test)**.** Let $T$ be a consistent $\kappa$-categorical theory with no finite models. Then $T$ is complete.

*Proof.* Let $M, N$ be models of $T$. Using Löwenheim-Skolem, we can find $M', N'$ with $\mathrm{Th}(M') = \mathrm{Th}(M)$ and $\mathrm{Th}(N') = \mathrm{Th}(N)$ such that $|M'| = |N'| = \kappa$. Then $M' \cong N'$, hence $M \equiv N$. $\square$

**Theorem 3.10** (Tarski's test)**.** If $\mathcal{B}$ is an $\mathcal{L}$-structure with domain $B$ and $A \subseteq B$. Then $A$ is the domain of an elementary substructure of $B$ iff for every formula $\varphi(x)$ with $\mathcal{B} \models \varphi(b)$ for some $b \in B$, we have $\mathcal{B} \models \varphi(a)$ for some $a \in A$.

**Definition 3.11.** Let $\mathcal{L}$ be a language (not necessarily relational). Let $T$ be an $\mathcal{L}$-theory. Let $\Sigma(x)$ be a set of forumlae. We say that $\Sigma(x)$ is **isolated mod $T$** iff three exists an $\mathcal{L}$-formula $\varphi(x)$ such that

- $T \cup \{\varphi(x)\}$ is consistent.
- For $\sigma(x) \in \Sigma(x)$, $T \models \forall x.(\varphi(x) \to \sigma(x))$.

**Example 3.12.** Let $T$ be a complete theory. Assume that $\Sigma(x)$ is isolated mod $T$. Then every model of $T$ realizes $\Sigma(x)$, i.e. $(\exists x)\varphi(x) \in T$.

A key tool for Theorem 4.12 is:

**Theorem 3.13** (Omitting types theorem)**.** Let $\mathcal{L}$ be countable. Let $T$ be a consistent theory. Let $\Sigma(x)$ be a set of formulae not isolated mod $T$. Then $T$ has a model not realising ("**omitting**") $\Sigma(x)$.

*Proof.* Pick a countably infinite set $\mathcal{C}$ of new constants. We'll extend $T$ to an $\mathcal{L}(\mathcal{C})$-theory $T^*$ such that

(a) $T^*$ is a Henkin theory, i.e. for every $\mathcal{L}(\mathcal{C})$ formula $\varphi(x)$, there is $c \in \mathcal{C}$ such that $(\exists x.\ \varphi(x)) \to \varphi(c)) \in T^*$.

(b) For all $c \in \mathcal{C}$, there is $\sigma(x) \in \Sigma(x)$ with $\neg\sigma(c) \in T^*$.

We construct $T^*$ inductively as the union of an increasing chain of consistent theories $T = T_0 \subseteq T_1 \subseteq \dots$. At each stage $T_n \subseteq T_{n+1}$ we add finitely many $\mathcal{L}(\mathcal{C})$-sentences. Enumerate $\mathcal{C} = \{c_i : i < \omega\}$ and enumerate all $\mathcal{L}(\mathcal{C})$-formulae in $x$ as $\{\psi_i(x) : i\omega\}$.

Given $T_{2i}$, take a constant $c \in \mathcal{C}$ not occurring in $T_{2i} \cup \{\psi_i(x)\}$. Let $T_{2i+1} :=$ $T_{2i} \cup \{(\exists x.\ \psi_i(x)) \to \psi_i(c)\}$.

Given $T_{2i+1} = T \cup T$, let $\delta(c_i, \overline{c}) = \bigwedge F$, $\overline{c} \subseteq \mathcal{C}$ $\overline{c} \not\ni c_i$. Then $\delta(x, \overline{y})$ is an $\mathcal{L}(\mathcal{C})$-formula and

$$\exists \overline{y} d(x, \overline{y})$$

is consistent with $T$, but it does not isolate $\Sigma(x)$.

So there exists $\sigma(x) \in \Sigma(x)$ and a model $M \models T$ with $M \models (\exists x)\,.\,(\exists y.\ \delta(x, \overline{y}) \wedge \neg\sigma(x))$.

Let $T_{2i+2}. = T_{2i+1} \cup \{\neg\sigma(c_i)\}$. Then $T_{2i+2}$ is consistent.

Let $(\mathcal{B}, (a_c)_{c \in \mathcal{C}}) \models T^*$.

Let $A := \{a_c : c \in \mathcal{C}\}$. By Tarski's Test (3.10) and since $T^*$ is Henkin, this is an elementary substructure of $\mathcal{B}$, □

# 4 $\omega$-Categoricity and Ryll-Nardzewski

$4^{\text{th}}$ talk, Prof. Hils, 2024-05-07

**Question 4.0.5.** What is special about $\aleph_0$ with respect to categoricity?[a]

[a] The name $\omega$-categoricity is used for historical reasons, but of course "$\aleph_0$-categoricity" would be a better name.

Motivation:

**Example 4.1.** Let $\mathcal{L} = \{f, P\}$, where $f$ is a unary function and $P$ a unary predicate. Let $T$ be the theory saying that models are infinite, $f \circ f = \mathrm{id}$ and $\forall x.\ x \in P \iff f(x) \notin P$.

This is $\kappa$-categorical for every $\kappa \geqslant \aleph_0$. Let $\mathcal{L}' := \{P\}$. Let $T' := T|_{\mathcal{L}'}$.

$T'$ is $\omega$-categorical, but not $\kappa$-categorical for $\kappa > \aleph_0$.

Note that $\omega$-categoricity passes to reducts, but this is not the case for $\kappa > \omega$.

**Definition 4.2.** A **complete $n$-type** in $T$, $p(x_1, \ldots, x_n)$ is a maximal finitely satisfiable set of $\mathcal{L}$-formulas $\varphi(x_1, \ldots, x_n)$.

Let $S_n(T)$ be the set of all complete $n$-types.

Let $A \subseteq \mathcal{M} \models T$, $T_A := \mathrm{Th}(\mathcal{M}_A)$. Let $S_n(T_A) =: S_n(A)$.

(With this notation we also write $S_n(\varnothing) := S_n(T)$).

**Example 4.3.** For $\overline{b} \in \mathcal{M}^n$ and $A \subseteq \mathcal{M}$, we set

$$\mathrm{tp}^m(\overline{b}/A) := \{\varphi(\overline{x})\ \mathcal{L}_A\text{-formula} : S_n(A) \ni M \models \varphi[\overline{b}]\}.$$

Any $p \in S_n(A)$ is of this form: By compactness, there exists $\mathcal{N} \succeq \mathcal{M}$, such that $\exists \overline{b} \in \mathcal{N}^n.\ \mathrm{tp}(\overline{b}/A) = p$.

**Definition 4.4.** The **stone topology** on $S_n(T)$ is the topology with basis of open sets given by $\langle\varphi\rangle \subseteq S_n(T)$, for $\varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula and $\langle\varphi\rangle := \{p : \varphi \in p\}$.

**Fact 4.4.6.** $S_n(T)$ is compact, Hausdorff and totally disconnected. (It even is profinite)

**Remark 4.4.7.**

(1) $p \in S_n(T)$ is **isolated** iff there exists $\varphi$, such that $\langle\varphi\rangle = \{p\}$. This is equivalent to $p$ being (topologically) isolated in $S_n(T)$.

(2) Exercise: If $\overline{a}, \overline{b}$ are finite tuples from $\mathcal{M} \models T$, then $\mathrm{tp}(\overline{a}, \overline{b})$ is isolated iff $\mathrm{tp}(\overline{b})$ and $\mathrm{tp}(\overline{a}/\overline{b})$ are both isolated.

**Definition 4.5.**

- $\mathcal{M}$ is **atomic** iff it only realizes isolated $n$-types for all $n \geqslant 0$.
- $\mathcal{M}$ is **$\omega$-saturated** iff for all $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ all $p \in S_n(A)$ are realized in $\mathcal{M}$.

  (equivalently this holds for all $p \in S_1(A)$).

**Fact 4.5.8.** For every $\mathcal{M} \models T$ there exists an elementary extension $\mathcal{N} \succeq \mathcal{M}$ which is $\omega$-saturated.

*Proof.* Catch your own tail: Take $\mathcal{M}_1 \succeq \mathcal{M}$ realizing all $p \in S_1(A)$ for all $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ (this exists by compactness) and similarly $\mathcal{M}_{n+1} \succeq \mathcal{M}_n$.

Let $\mathcal{N} := \bigcup_{i<\omega} \mathcal{M}_i$. □

**Definition 4.6.** If

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \cap\!\!| & & \cap\!\!| \\ \mathcal{M} & & \mathcal{M}' \end{array}$$

where $\mathcal{M}, \mathcal{M}' \models T$, $f$ is called a **partial elementary** embedding if it preserves formulas with variables form $A$, i.e.

$$\mathcal{M} \models \varphi(\overline{a}) \iff \mathcal{M}' \models \varphi(f(\overline{a}))$$

for all $\overline{a} \in A^n$, i.e. $\mathrm{tp}(A) = \mathrm{tp}(f(A)) = \mathrm{tp}(A')$.

**Definition 4.7.** $\mathcal{M}$ countable is called **homogeneous**iff every partial elementary $f\colon A \to B$, $A, B \overset{\text{finite}}{\subseteq} \mathcal{M}$ extends to $\sigma \in \mathrm{Aut}(\mathcal{M})$.

**Proposition 4.8.**

(1) Let $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M}$ countable atomic.

Then any $f\colon A \to \mathcal{N}$ partial elementary with $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ extends to an elementary embedding $F : \mathcal{M} \overset{\preceq}{\hookrightarrow} \mathcal{N}$.

(2) If $\mathcal{M}$ and $\mathcal{N}$ are both atomic then any $f$ as in (1) extends to an isomorphism. In fact the atomic countable modle is unique if it exists.

(3) $\mathcal{M}$ countable atomic $\implies$ $\mathcal{M}$ homogeneous.

*Proof.* Let $f\colon A \to \mathcal{N}$. Let $\mathcal{M} = \{a_i | i \in \mathbb{N}\}$. Given $f_k : \{a_0, \dots, a_h\} \to \mathcal{N}$ partial elementary. Consider $\operatorname{tp}(a_{h+1}/a_{\leqslant h}))$ this is isolated by assumption. Hence $f_{h*}$ is an isolated type in $S_1(\operatorname{im}(f_h))$. Let $b_{h+1}$ be a realization, $a_{h+1} \mapsto b_{h+1}$.

(2) is a symmetric version of this and (3) is a special case of (2). □

**Proposition 4.9.**

(1) Let $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M}$ countable, $\mathcal{N}$ $\omega$-saturated. Then every $f\colon A \to \mathcal{N}$ partial elementary, where $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ extends to $\tilde{f}\colon \mathcal{M} \overset{\preceq}{\hookrightarrow} \mathcal{N}$.

(2) Let $\mathcal{M}, \mathcal{N}$ be countable and $\omega$-saturated. Then every $f$ as in (1) extends to an isomorphism.

In particular, a countable $\omega$-saturated model is unique (if it exists).

(3) If $\mathcal{M}$ is countable and $\omega$-saturated, then $\mathcal{M}$ is homogeneous.

*Proof.* (1) is basically the same as before using $\omega$-saturation of $\mathcal{N}$.

Again (2) is a symmetric version and (3) a special case. □

**Example 4.10.**

- Consider $\mathrm{ACF}_0$. $\mathbb{Q}^{\text{alg}}$ is atomic. Let $\{t_i | i \in \mathbb{N}\}$ be algebraically independent. Then $\mathbb{Q}(t_i | i \in \mathbb{N})$ is $\omega$-saturated.
- Consider infinite $\mathbb{Q}$-vector spaces. $\mathbb{Q}$ is atomic. $\mathbb{Q}^{(\mathbb{N})}$ is $\omega$-saturated.
- Graphs: $\mathbb{Z}$ is atomic, $\coprod_{i<\omega} \mathbb{Z}$ is $\omega$-saturated.
- DOAG[a] $= \operatorname{Th}(\mathbb{Q}, +, -, \leqslant, 0)$: $\mathbb{Q}$ is atomic. However no $\omega$-saturated countable model exists, since $|S_2(T)| = 2^{\aleph_0}$ (reals can be coded in 2-types).

[a] divisible ordered abelian groups

**Observe.** *If $G \curvearrowright X$, we get $G \curvearrowright X^n$ via $g.(x_1, \dots, x_n) \coloneqq (gx_1, \dots, gx_n)$.*

**Definition 4.11.** A group action $G \curvearrowright X$ is called **oligomorphic** if for every $n$, $G \curvearrowright X^n$ has finitely many orbits.

**Observe.**[5] *$X \subseteq \mathcal{M}^n$ is* $\mathrm{Aut}(\mathcal{M})$*-invariant iff it is the union of* $\mathrm{Aut}(\mathcal{M})$*-orbits.*

**Theorem 4.12** (Ryll-Nardzewski, Engler, Svenonius)**.** Let $T$, $\mathcal{L}$ as above.

The following are equivalent:

(1) $T$ is $\omega$-categorical.

(2) Any $p \in S_n(T)$ is isolated for all $n \in \mathbb{N}$.

(3) $S_n(T)$ is finite for all $n$.

(4) There are infinitely many $\mathcal{L}$-formulae $\varphi(x_1, \ldots, x_n)$ up to $\sim_T$.

(5) There exists $\mathcal{M} \models T$ countable, such that $\mathrm{Aut}(\mathcal{M}) \curvearrowright \mathcal{M}$ is oligomorphic.

(6) There exists $\mathcal{M} \models T$ countable such that all $\mathrm{Aut}(\mathcal{M})$-invariant $X \subseteq \mathcal{M}^n$ are $\emptyset$-definable (i.e. $X = \varphi[M]$).

(7) There exists $\mathcal{M} \models T$ countable such that $\mathcal{M}$ only realizes finitely many $n$-types for all $n \in \mathbb{N}$.

*Proof.* (1) $\implies$ (2): Suppose not, i.e. $p \in S_n(T)$ is not isolated. Then there exists a countable model $\mathcal{M} \models T$ realizing $p$ (Löwenheim-Skolem, compactness). By the Omitting Types Theorem (3.13), there exists $\mathcal{N} \models T$ countable omitting $p$. $\mathcal{M} \cong \mathcal{N} \lightning$.

(2) $\iff$ (3) $\iff$ (4) are obviously equivalent.

Assume (2). For $\mathcal{M} \models T$ countable, we then have $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$, $\overline{a}, \overline{b} \in \mathcal{M}^n$ iff there exists $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma(\overline{a}) = \overline{b}$. (here we use that $\mathcal{M}$ is atomic and hence homogeneous). So $\mathrm{Aut}(\mathcal{M})$-orbits in $\mathcal{M}^n$ correspond to $n$-types realized in $\mathcal{M}$. We get (2) $\implies$ (5): By (2) $\implies$ (3), we get that $S_n(T)$ is finite.

(2) $\implies$ (6): By (2) $\implies$ (3), every Aut-invariant subset $X \subseteq \mathcal{M}^n$ is a finite union of realizations of isolated types.

(6) $\implies$ (7): Let $\mathcal{M}$ be as in (6). Suppose $\mathcal{N}$ realizes infinitely many $n$-types $p_1, p_2, \ldots$ For $I \subseteq \mathcal{N}$ let $Y_I := \bigcup_{i \in I} p_i[\mathcal{M}]$. This gives rise to $2^{\aleph_0}$ invariant subsets. But $\mathcal{L}$ is countable.

(7) $\implies$ (3):

Set set of types $p \in S_n(T)$ realized in $\mathcal{M}$ is dense in $S_n(T)$. So everything, by finiteness. (5) $\implies$ (7): trivial.

(2) $\implies$ (1): Countable atomic models are unique by the proposition. $\square$

[5] or use this as a definition.

**Example 4.13.** (1) Let $\mathcal{L}_{\mathrm{gp}} = \{e, \circ\}$ be the language of groups and $G$ infinite and $\omega$-categorical. Then there exists $n$ such that $g^n = e$ for all $g \in G$, i.e. $G$ is of finite exponent.

Indeed, there can be only finitely many finite orders. There exists no element $g$ of infinite order, as otherwise, $(g, g^n)$ would be $n$ distinct 2-types.

(2) No infinite field is $\omega$-categorical.

By (1), characteristic 0 is impossible. If $K$ is infinite, then then for all $n \in \mathbb{N}$, there exists $x \in K^*$ with $x^n \neq 1$.

(3) If $\mathcal{L}$ is a finite language without function symbols, and $T$ a complete $\mathcal{L}$-theory with infinite models and quantifier elimination, then $T$ is $\omega$-categorical.

E.g. an ultrahomogeneous structure in a finite language without function symbols. E.g. $\mathrm{Fr}(K)$, where $K$ is a Fraïssé class in $\mathcal{L}$ as before.

**Corollary 4.14.** If $T$ is an $\omega$-categorical $\mathcal{L}$-theory and $\mathcal{L}' \subseteq \mathcal{L}$, then $T' := T|_{\mathcal{L}'}$ is $\omega$-categorical.

# 5 Ryll-Nardzewski II

$5^{\text{th}}$ talk, Martin Hils, 2024-05-14

Let $\mathcal{L}$ be countable.

Recall that if $\mathcal{M}$ is a countable, ultrahomogeneous $\mathcal{L}$-structure in a finite language without function symbols, then $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical.

Indeed, $\mathrm{Aut}(\mathcal{M}) \rightharpoondown \mathcal{M}$ is oligomorphic in this case.

Now we want to consider languages that also have function symbols:

**Definition 5.1.** Let $\mathcal{L}$ be finite (possibly including function symbols). A class $\mathcal{K}$ of finitely generated $\mathcal{L}$-structurs is called **uniformly locally finite** if there exists $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $A \in \mathcal{K}$ generated by $n$ elements we have $|A| \leqslant f(n)$.

An infinite $\mathcal{L}$-structure $\mathcal{M}$ is called **uniformly locally finite** if $\mathrm{Age}(\mathcal{M})$ is.

Recall from last week: If $G$ is an $\omega$-categorical group, then $G$ is of finite exponent. By the same argument we get the following:

**Lemma 5.2.** Let $\mathcal{L}$ be a countable language. If $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical, then $\mathcal{M}$ is uniformly locally finite.

Recall:

**Definition 5.3.** If $A \subseteq \mathcal{M}$, then the model-theoretic algebraic closure is defined
$$\operatorname{acl}(\mathcal{M}) = \bigcup_{\substack{\varphi \mathcal{L}_A\text{-formula} \\ \varphi[M] \text{ finite}}} \varphi[M].$$

We have $A \subseteq \langle A \rangle \subseteq \operatorname{dcl}(A) \subseteq \operatorname{acl}(A) = \operatorname{acl}(\operatorname{acl}(A))$.

**Example 5.4.** In vector spaces, this is the linear span. In ACF it is the field theoretic algebraic closure, hence the name.

**Definition 5.5.** $\mathcal{M}$ is said to have **uniformly finite algebraic closure** iff there exists $g \colon \mathbb{N} \to \mathbb{N}$ such that for all $A \overset{\text{finite}}{\subseteq} \mathcal{M}$ we have $|\operatorname{acl}(A)| \leqslant g(|A|)$.

**Lemma 5.6** (Lemma 5.2'). If $\mathcal{M}$ is $\omega$-categorical, then $\mathcal{M}$ has uniformly finite algebraic closure.

*Proof.* If $\operatorname{tp}(a_1, \ldots, a_n) = \operatorname{tp}(b_1, \ldots, b_n)$, then $\operatorname{acl}(a_1, \ldots, a_n) \cong \operatorname{acl}(b_1, \ldots, b_n)$.

Hence uniform finiteness of acl follows from finiteness of $\operatorname{acl}(a_1, \ldots, a_n)$. Towards a contradiction assume that $\operatorname{acl}(a_1, \ldots, a_n) =: \tilde{A}$ is infinite, say $\tilde{A} = (b_i)_{i \in \mathbb{N}}$. If $\operatorname{tp}(a_1, \ldots, a_n, b_j) = \operatorname{tp}(a_1, \ldots, a_n, b_k)$, there exists an $\mathcal{L}_A$-formula $\varphi(x)$ with $\varphi[M]$ finite, such that $b_j, b_k \in \varphi[M]$. Hence there are infinitely many different types of the form $\operatorname{tp}(a_1, \ldots, a_n, b_j)$ $\lightning$. ☐

Recall:

**Definition 5.7.** $T$ has **quantifier elimination** (QE) if every $\mathcal{L}$-formula $\varphi(X_1, \ldots, X_n)$ is equivalent modulo $T$ to a quantifier free formula.

**Proposition 5.8.** Suppose that $\mathcal{L}$ is finite an $\mathcal{M}$ a countably infinite $\mathcal{L}$-structure. Then the following are equivalent:

(1) $\mathcal{M}$ is ultrahomogeneous and uniformly locally finite.

(2) $\mathcal{M}$ is $\omega$-categorical and has QE.

*Proof.* (1) $\implies$ (2): Since there are only finitely many $n$-generated substructures, there are only finitely many orbits of $\operatorname{Aut}(\mathcal{M}) \rightharpoonup \mathcal{M}$ by ultrahomogeneity.

Since $\operatorname{tp}(\overline{a}) = \operatorname{tp}(\overline{b}) \iff \langle \overline{a} \rangle = \langle \overline{b} \rangle$, we get that $\{\operatorname{tp}(\overline{a})\} = \langle \varphi \rangle$ for some quantifier free formula $\varphi$.

Every $\mathcal{L}$-formula is equivalent modulo $T$ to a finite disjunction of such formulae.

Alternative proof of QE:

**Fact 5.8.9.** Let $T$ be a theory such that for all $\mathcal{M}, \mathcal{N} \models T$, $A \subseteq \mathcal{M}, \mathcal{N}$ a finitely generated common substructure, $\mathcal{N}$ $\omega$-saturated, $b \in \mathcal{M}$, there is $b' \in \mathcal{N}$ such that $\langle A, b \rangle = \langle A, b' \rangle$ via $b \mapsto b'$.

(2) $\implies$ (1): We already know uniformly locally finite by Lemma 5.2 (this does not need QE).

A $\mathcal{M}$ is $\omega$-saturated (by $\omega$-categoricity) and we have QE, it follows that we have the extension property. This implies ultrahomogeneity. $\square$

*Proof (Alternative Proof).* Let $A$ be a finite $\mathcal{L}$-structure generated by $(a_1, \ldots, a_n)$. Then there is a quantifier free formula $\psi_{A,\overline{a}}(x_1, \ldots, x_n)$ (given by the simple diagram of $A$) such that for all $\overline{b} = (b_1, \ldots, b_n)$, $\overline{b} \in \mathcal{N}$, then $\mathcal{N} \models \psi_{A,\overline{a}}(\overline{b})$ iff $\overline{b} \mapsto \overline{a}$ defines an isomorphism $\langle \overline{b} \rangle \cong \langle \overline{a} \rangle$. Now if $\mathrm{Age}(\mathcal{M})$ is uniformly locally finite, there are only finitely many substructures of $\mathcal{M}$ generated by $n$ elements up to isomorphism.

- Let $U_0$ be the following set of $\mathcal{L}$-sentences:

  $$\forall \overline{x}.\ (\psi_{A,\overline{a}}(\overline{x}) \to \exists y.\ \psi_{B,\overline{a}^\frown b}(\overline{x}, y)),$$

  where $A = \langle a_1, \ldots, a_n \rangle \subseteq B = \langle a_1, \ldots, a_n, b \rangle$. For $n = 0$ take $\exists y.\ \psi_{B,b}(y)$.

- Let $U_1$ be the following set of sentences:

  $$\forall x_1, \ldots, x_n.\ \bigvee_{\substack{A,\overline{a} \\ |\overline{a}|=n}} \psi_{A,\overline{a}}(x_1, \ldots, x_n).$$

If $\mathcal{N} \models U_0 \cup U_1$ is countably infinite, then $\mathcal{N} \cong \mathcal{M}$ as $\mathcal{N}$ is the Fraïssé limit of $\mathrm{Age}(\mathcal{M}) = \mathrm{Age}(\mathcal{N})$.

$\square$

**Example 5.9.**

- Let $K$ be the class of all finite groups. $K$ is locally finite, but not uniformly locally finite. $K$ is a Fraïsse class (cf. section 2) $\mathrm{Fr}(K)$ is not $\omega$-categorical (since it is not of finite exponent). (it also does not have QE, but we don't want to check this).

- Let $K$ be the class of all finite abelian groups. Then $\mathrm{Fr}(K) = \bigoplus_{i \in \mathbb{N}} (\mathbb{Q}/\mathbb{Z})$. $\mathrm{Th}(\mathrm{Fr}(K))$ has QE (since it is divisible).

  We have $\bigoplus_{n \in \mathbb{N}} \mathbb{Q}/\mathbb{Z} \preceq \bigoplus_{n \in \mathbb{N}} \mathbb{Q}/\mathbb{Z} \oplus \bigotimes_{j \in J} \mathbb{Q}$, so it is not $\omega$-categorical. (We already knew this since it is not uniformly locally finite.)

- Fix $n \geqslant 2$. Let $K_n$ be the class of all finite abelian groups of exponent $n$. Exercise (easy): $K_n$ is a Fraïsse class. We have $\mathrm{Fr}(K_n) \cong \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/n =: A_n$.

This is $\omega$-categorical, but not always $\aleph_1$-categorical. It is $\aleph_1$-categorical iff $n = p^m$ for some prime $p$.

Let $n = p_1 \cdot \ldots \cdot p_l$. We have $A_n \geqslant p_1 A_n \geqslant p_1 p_2 A_n \geqslant \ldots \geqslant 0$. Hence $\mathrm{MR}\left(\bigoplus_{i\in\mathcal{N}} \mathbb{Z}/n\mathbb{Z}\right) \geqslant l$ (in fact equality holds).

- Generic BLF: a linear algebra "analogue" of the random graph.

  Fix a prime $p$. Consider the class $K$ of $(V, \mathbb{F}_p, \beta)$, where $V$ is a finite dimensional $\mathbb{F}_p$-vector space, $\beta\colon V^2 \to \mathbb{F}_p$ is a symmetric bilinear form.

  This is a Fraïssé class. JEP and HP are clear.

  Consider $\mathrm{Fr}(K) =: (V, \mathbb{F}_p, \beta) =: \mathcal{M}$. We know that $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical with QE.

  It satisfies the following axioms:

  – Symmetric bilinear form on $\mathbb{F}_p$-vector spaces.

  – Given $\lambda_1, \ldots, \lambda_n \in \mathbb{F}_p$, for all $x_1, \ldots, x_n \in V$, iff $x_1, \ldots, x_n$ are $\mathbb{F}_p$-linearly independent, then there exists $y$ such that $\bigwedge_{i=1}^n \beta(x_i, y) = \lambda_i$

  This already is a full axiomatization.

  Random graph and generic BLF have IP.

**Proposition 5.10** (Proposition 5.8'). Let $\mathcal{L}$ be finite and $\mathcal{M}$ countably infinite.

Then the following are equivalent:

(1) $\mathcal{M}$ has uniformly finite acl and any isomorphism between finite algebraically closed subsets of $\mathcal{M}$ extends to an automorphism $\sigma \in \mathrm{Aut}(\mathcal{M})$.

(2) $\mathrm{Th}(\mathcal{M})$ is $\omega$-categorical and every $\mathcal{L}$-formula is equivalent modulo $T$ to a boolean combination of bounded existential formulae. , where bounded existential means $\varphi(\overline{x}) := \exists \overline{y}.\ \psi(\overline{x}, \overline{y}$, where $\psi$ is quantifier free and such that $\mathcal{M} \models \psi(\overline{a}, \overline{b})$, then $\overline{b} \in \mathrm{acl}(\overline{a})$.

*Proof.* (2) $\implies$ (1) by Lemma 5.6 and the fact that the formulae of the form in question determine the complete type.

(1) $\implies$ (2): Write down axioms for the extension property of finite acl-closed sets. □

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