Stable Groups

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Contents

These are my notes on the lecture "Stable Groups", taught by Prof. Dr. Dr. Katrin Tent in the summer term 2024 at the University Münster.

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[Lecture 01, 2024-04-15]

A background in model theory is helpful but not necessary. Some group theory is required (usually covered in linear algebra and a first algebra course).

The lecture starts at 08:25.

The book by Prof. Tent is available on learnweb.

There will be an oral exam. For a type II course, one needs to do nothing.

The main point is to see, how model theoretic properties influence algebraic properties.

1 Introduction

Definition 1.1. An infinite $\mathcal{L}$-structure M is **minimal**, iff for every formula $\varphi(x) \in \mathcal{L}(M)$, the set defined by φ, $\varphi(M) := \{a \in M | M \models \varphi(a)\}$ is finite or cofinite.

Example 1.2. $(\mathbb{Q}, +, \cdot)$ is not minimal, consider or example the formula $\varphi(x) := \exists y.\ x = y^2$

$(\mathbb{C}, +, \cdot)$ is minimal.

Recall the **orbit equation**: If $G \curvearrowright X$ is transitive, then there is a natural bijection

$$\begin{aligned} {}^{G}\!/\!_{G_x} &\longrightarrow X \\ hG_x &\longmapsto h \cdot x \end{aligned}$$

where for $x \in X$, $G_x := \{g \in G : gx = x\} \leqslant G$ is the **stabilizer** of x in G and $G \cdot x := \{gx : g \in G\} \subseteq X$ is the **orbit** of x under G.

Theorem 1.3 (Reineke)**.** Minimal groups are abelian.

Proof. Let G be a minimal group.

Since G is minimal, all proper definable[1] subgroups are finite by minimality: If $H \lneq G$ is a proper definable subgroup, then for $a \notin H$, the coset $a \cdot H$ is also definable and disjoint from H.

Suppose that G is not abelian. Then the center[2] $Z(G)$ is finite. Furthermore, every element of the group must have finite order, since $\langle a \rangle \leqslant Z(\mathrm{Cen}(a))$.[3] (Note that $\langle a \rangle$ is not definable in general).

[1] A **definable** subgroup is a subgroup, that can be defined by a formula.
[2] The **center** is defined as $Z(G) := \{x \in G : \forall y.\ xy = yx\}$.
[3] The **centralizer** of a is the set of all elements commuting with a.

Consider the conjugacy class $a^G := \{a^g : g \in G\}$, where $a^g := g^{-1}ag$. Then for $a \in Z(G)$,[4] we have $|a^G| = |{}^G\!/\!_{G_x}|$, where $G_a = \mathrm{Cen}(a)$. In particular, for $a \notin Z(G)$, the conjugacy class a^G is infinite. Since by minimality there can not be two disjoint infinite conjugacy classes, we get $G = a^G \cup Z(G)$ for all $a \notin Z(G)$. Thus any $a, b \in G \backslash Z(G)$ are conjugate, so a, b have the same finite order and $|\mathrm{Cen}(a)| = |\mathrm{Cen}(b)|$.

If all elements have order 2, the group is abelian, since $a^{-1}b^{-1}ab = abab = 1$ in this case.

If all $a \in G \backslash Z(G)$ have order 2, then again G is abelian: Let $c \in Z(G)$, then $ca \notin Z(G)$, so $1 = (ac)^2 = acac = a^2c^2 = c^2$, i.e. the elements in $Z(G)$ also have order 2.

Now let $a \in G \backslash Z(G)$. Then $a^2 \neq 1$ and $a, a^{-1} \notin Z(G)$ are conjugate under some $g \in G$, i.e. $b^{-1}ab = a^{-1}$, hence $b^{-2}ab^2 = a$, $b^2 \in \mathrm{Cen}(a)$. So $a \in \mathrm{Cen}(b^2) \backslash \mathrm{Cen}(b)$. Clearly $\mathrm{Cen}(b) \leqslant \mathrm{Cen}(b^2)$ and a witnesses that this is a proper subgroup. So $|\mathrm{Cen}(b)| \neq |\mathrm{Cen}(b^2)|$, hence $b^2 \in Z(G)$. It follows that $H = {}^G\!/\!_{Z(G)}$ is an elementary abelian 2-group in which all non-trivial elements are conjugate, i.e. $|H| = 2$[5] and so G is finite. □

We want to generalize this.

Definition 1.4. An $\mathcal{L}$-structure M is **stable** iff there are no $M \preceq \tilde{M}$,[a] $\mathcal{L}(\tilde{M})$-formula $\varphi(\overline{x}, \overline{y})$ and tuples $\overline{a_i}, \overline{b_j} \in \tilde{M}$ such that $\tilde{M} \models \varphi(\overline{a_i}, \overline{b_j})$ iff $i < j$.

[a] elementary extension

Example 1.5. Let $M = (\mathbb{Z}, +, \cdot, 0, 1)$, $a_i = i = b_i$ and $\varphi(x, y) \longleftrightarrow \ulcorner \exists z_1, \ldots, z_4.\ x + z_1^2 + \ldots + z_4^2 = y \urcorner$. Then $M \models \varphi(a, b_j)$ iff $i \leqslant j$. So M is not stable.

Algebraically closed fields are stable.

Lemma 1.6. If M is a stable and non-empty **semigroup**[a] with right- and **left-cancellation**[b] (alternatively: left-cancellation and a **right neutral element**[c]), then M is abelian.

[a] associative operation
[b] $ax = ay \implies x = y$
[c] $\forall a.\ ae = a$.

Proof. The formula $\varphi(x, y) \longleftrightarrow \ulcorner \exists z.\ x \cdot z = y \urcorner$, is satisfied by (a^n, a^m) if $n < m$. By stability, this can not be an if and only if. So there must be some $m > n$,

[4] Note that for $a \in Z(G)$, we have $a^G = \{a\}$.
[5] Conjugation is not too interesting in abelian groups.

such that $M \models \varphi(a^n, a^m)$. I.e. there is some $b \in M$ such that $a^n = a^{n+p}b$, where $m = n + p$. Put $e = a^p b$. This is a left-neutral element: For $c \in M$ we have $a^n c = a^n ec$, hence $c = ec$ by left-cancellation.

By symmetry (or assumption), there exists a right-neutral element f, and since $e = ef = f$, e is neutral.

Furthermore

$$e = a^p b \overset{p \geq 0}{=} a(a^{p-1}b),$$

so a has an inverse. □

Remark 1.6.1. The assumptions are necessary since a semigroup with $xy = y$ is not a group.

Corollary 1.7. If G is stable, then every non-empty definable subset closed under multiplication is a subgroup.

Similarly, every definable non-empty subring of a stable field is a subfield.

Remark 1.7.2. A stable group is a group whose theory is stable, (not necessarily in the language of groups). The group may be a (or interpretable) structure inside another structure, e.g. $(K, +, \cdot, 0, 1)$ field, $G = \mathrm{GL}_n(K)$ or any other Chevalley group.

Definition 1.8. A definable group action (in some L-structure M) is given by a definable group G, a definable set X and a definable action $G \times X \to X$ (i.e. the graph of the action is a definable subset of $(G \times X) \times X$.

Example 1.9. Let $(K, +, \cdot, 0, 1)$ be a field. Then $\mathrm{GL}_n(K)$, K^n and $\mathrm{GL}_n(K) \rightharpoondown K^n$ are definable.

Example 1.10. Consider $(\mathbb{Q}, +, \cdot, 0, 1)$. Then $A := [0, 1]$ is definable[a] and $\frac{1}{n}A \subsetneq A$. Hence it is not stable by the following lemma.

[a] this is non-trivial

Lemma 1.11. Let G be a stable group acting definably on a set X. If $A \subseteq X$ is definable and $g \in G$, then $g(A) \subseteq A$ iff $g(A) = A$.

Proof. If $g(A) \subsetneq A$, we get a proper descending sequence $A \supsetneq g(A) \supsetneq g^2(A) \supsetneq g^3(A) \supsetneq \dots$ and the sequence g^i, $i < \omega$ is ordered by $\ulcorner xA \subsetneq yA \urcorner$. □

[Lecture 02, 2024-04-18]

Recall:

Corollary 1.12 (of Lemma 1.11)**.** If G is stable, $A \subseteq G$ is definable and $g \in G$, then $A^g \leqslant A \iff A^g = A$.

Remark 1.12.3. This does not hold in general. Consider

$$H = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\} < \mathrm{GL}_2(\mathbb{Q})$$

and

$$g := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, h_m := \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}.$$

Then $gh_m g^{-1} = h_{2m}$, so $gHg^{-1} \lneq H$.

Definition 1.13. For a group G, a family $\{H_i\}_{i \in I}$ of subsets $\{H_i\}_{i \in I}$ of M^k is called **uniformly definable** if there is a formula $\varphi(\overline{x}, \overline{y})$ and $\overline{a_i} \in M_i, i \in I$ such that $\varphi(M^k, \overline{a_i}) = H_i$.

For example, the centralizers of elements are uniformly definable.

Remark 1.13.4. If G is stable, the **Trivial Chain Condition** holds for uniformly definable subsets and subgroups, i.e. descending chains are finite:

For every uniformly definable family $H_i, i \in I$, there is some $n < \omega$ such that every properly descending (resp. ascending) chain $H_{i_1} \lneq H_{i_2} \lneq H_{i_3} \lneq \ldots$ has length at most n. This n depends only on the formula, not on the parameters of the form of the definable family.

Definition 1.14. A formula $\varphi(\overline{x}, \overline{y})$ has the **independence property** (**IP**) iff there are $\overline{a_i}, i < \omega$, such that for all $A \subseteq \omega$, the set $\{\varphi(\overline{x}, \overline{a_i}) | i \in A\} \cup \{\neg\varphi(\overline{x}, \overline{a_i}) | i \notin A\}$ is consistent.

relate to wikipedia definition (use compactness), see proof of Lemma 1.17

A theory is called **NIP** iff no consistent formula has IP.

Example 1.15. The random graph (Radograph) has IP. $(\mathbb{C}, +, \cdot, 0, 1)$ is NIP.

Lemma 1.16. If T is stable, then T is NIP.

Proof. If $\varphi(\overline{x}, \overline{y})$ has IP, $\overline{a_i} \in M, i < \omega$, $M \models T$, then there are $\tilde{M} \succeq M$, and $b_i \in \tilde{M}$ such that $\tilde{M} \models \varphi(\overline{b_i}, \overline{a_j})$ iff $i < j$, which is a contradiction to stability. $\square$

The reverse direction does not hold, since for example the real numbers have

NIP, but are not stable.

Lemma 1.17. Let G be a NIP group. Then finite intersections of uniformly definable subgroups are uniformly bounded, i.e. for every formula $\varphi(x, \overline{y})$ there is $n < \omega$ such that if $H_i = \varphi(G, \overline{a_i})$, $i = 1, \ldots, m$ are subgroups, then

$$\bigcap_{i \leqslant m} H_i = \bigcap_{j=1}^{n} H_{i_j}.$$

Proof. Suppose not. Then for all $n < \omega$ there is a uniformly definable family of subgroups $H_1, \ldots, H_n$ such that $\bigcap H_i \lneqq \bigcap_{\substack{i=1 \\ i \neq j}}^{n} H_i$ for any $1 \leqslant j \leqslant n$.

So there is some $b_j \in \left(\bigcap_{i \neq j} H_i\right) \backslash H_j$, $j \leqslant n$.

For $I \subseteq \{1, \ldots, n\}$ put $b_I := \prod_{i \in I} b_i$. Then $G \models \varphi(b_j, \overline{a_i})$ iff $i \notin J$. Since n was arbitrary, this shows that $\varphi(x, y)$ has IP: Let $A \subseteq \omega$ be any subset. By the compactness theorem it suffices to show that every finite subset of $\{\varphi(x, \overline{a_i}) | i \in A\} \cup \{\neg\varphi(x, \overline{a_i}) | i \notin A\}$ is consistent. This holds, since for every $I \overset{\text{finite}}{\subseteq} \omega$,

$$G \models \{\varphi(b_{I \backslash A}, \overline{a_i}) | i \in A \cap I\} \cup \{\neg\varphi(b_{I \backslash A}, \overline{a_i}) | i \in I \backslash A\}.$$

□

Proposition 1.18 (Baldwin-Saxl)**.** If G is stable, then for every formula $\varphi(\overline{x}, \overline{y})$, there is $n < \omega$ (depending only on φ) such that for subgroups $H_i = \varphi(G, \overline{a_i})_{i \in I}$, we have

$$\bigcap_{i \in I} H_i = \bigcap_{j=1}^{n} H_{i_j}$$

for some $i_j \in I$, i.e. arbitrary intersections of uniformly definable subgroups are definable.

Proof. By Lemma 1.17 intersections of finitely many H_i are uniformly definable. By the Trivial Chain Condition (1.13.4) applied to these uniformly definable intersections, there is a minimal group H is this family, i.e. $H = \bigcap H_i = \bigcap_{j=1}^{n} H_{i_j}$ with $n < \omega$ from Lemma 1.17. □

Corollary 1.19. If G is stable and $A \leqslant G$ arbitrary, then $\mathrm{Cen}(A) = \bigcap_{a \in A} \mathrm{Cen}(a) = \{g \in G | \forall a \in A.\ [g, a] = 1\}$ is definable.

Example 1.20. By Sela's Theorem the free groups F_n are stable. For $w \in F_k$, $\mathrm{Cen}(w)$ is cyclic, so $n = 2$.

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Remark 1.20.5. Since the formula $\ulcorner xa = ax \urcorner$ is quantifier-free, Corollary of Baldwin-Saxl (1.19) holds in all subgroups of stable groups.

For example $\mathrm{Sym}_{\mathrm{fin}}(\omega)$, the group of permutations of ω with finite support (i.e. moving only finitely many elements) can never be a subgroup of a stable group, since centralizers can become arbitrarily small.

Definition 1.21. Let T be arbitrary and $\varphi_s(\overline{x}), s \in 2^{<\omega}$ consistent formulae.[a]

Then

(i) the $\varphi_s(\overline{x})$ form a **binary tree of consistent formulae** iff

$$T \vdash \forall \overline{x}(\varphi_{s^\frown 0}(\overline{x}) \vee \varphi_{s^\frown 1}(\overline{x}) \to \varphi_s(\overline{x}))$$

and

$$T \vdash \forall \overline{x} \neg(\varphi_{s^\frown 0}(\overline{x}) \wedge \varphi_{s^\frown 1}(\overline{x})).$$

(ii) T is called **totally transcendental** (or **ω-stable** iff $\mathcal{L}$ is countable) iff there is no binary tree of consistent formulae.

[a] Here "consistent" means that the family is consistent along every path, i.e. for every $s \in 2^\omega$, $\{\varphi_{s|_n} : n \in \omega\}$ is consistent. The entire family may be inconsistent.

Example 1.22. Let G be a group, and $H_i, i < \omega$ an infinite descending chain of subgroups $H_i \gneq H_{i+1}$, then we get a binary tree (subset vs. coset). So totally transcendental is much stronger than stable.

Proposition 1.23. If G is totally transcendental, there is no infinite properly descending chain of definable subgroups.

Proof. Otherwise we get a binary tree.

□

[Lecture 03, 2024-04-22]

Corollary 1.24.

(i) In a totally transcendental group G every intersection of definable subgroups is definable. In particular, there is a minimal definable subgroup G^0 of finite index in G, the **connected component** of G.

(ii) If G is totally transcendental, every injective definable endomorphism of G is surjective, i.e. an automorphism of G.

(iii) If G is ω-stable, abelian and torsion free, then G is divisible[a].

[a]An abelian group A is **divisible** iff $\forall a \in A.\ \forall n \in \mathbb{N}.\ \exists b \in A.\ n \cdot b = a$, i.e. iff $G \cong \bigotimes_{i\in I}\mathbb{Q}$.

Proof. (i) Clear.

(ii) Suppose that $s\colon G \hookrightarrow G$ is definable but not surjective. Then $s^i(G)$ is a proper descending sequence of definable subgroups $\lightning$.

(iii) Note that the map $g \mapsto n \cdot g$ is definable and injective.[6]

□

Remark 1.24.6. If G is stable, then for any formula $\varphi(x, \overline{y})$ the group

$$G^0(\varphi) = \bigcap\{\varphi(G, \overline{a}) | \varphi(G, \overline{a}) \leqslant G, [G : \varphi(G, \overline{a})] < \infty\}$$

is a definable subgroup of finite index by Baldwin-Saxl (1.18), the **φ-connected component of G**.

In particular, we'll be interested in the case

$$\varphi(x, y) \longleftarrow \ulcorner xy = yx \urcorner.$$

Definition 1.25. A group G is called **centralizer connected** iff $G = G^0(xy = yx)$, i.e. iff for all $a \in G\backslash Z(G)$ the index $[G : \mathrm{Cen}(a)]$ is infinite.

Lemma 1.26. If G is centralizer connected, $A \subseteq G$ finite and A normalized by[a] G, then $A \subseteq Z(G)$.

[a]For $A, B \leqslant G$ we say that A is **normalized** by B iff $\forall b \in B.\ A^b = A$, i.e. $B \leqslant N_G(A)$.

Proof. If a^G is finite, then $a \in Z(G)$, since $|G : \mathrm{Cen}(a)| = |a^G|$. □

Remark 1.26.7. This does not depend on stability.

Proposition 1.27. If G is stable[a] and $\{[g, h] | g, h \in G\}$ finite, then G is virtually abelian.[b]

[a]The assumption of G being stable can be removed.

[6]Warning: $g \mapsto n \cdot g$ is not uniformly definable.

[b] A group is called **virtually abelian** or **abelian-by-finite** iff $Z(G)$ has finite index in G.

Proof. For every $g \in G$, the set $\{[g,h] : h \in G\}$ is finite. Hence g^G is finite, so $|G : \mathrm{Cen}(g)|$ is finite. By the Corollary of Baldwin-Saxl (1.19), we have $Z(G) = \bigcap_{i \leqslant n} \mathrm{Cen}(g_i)$ for some $n \in \mathbb{N}$, and this has finite index. ☐

Proposition 1.28. If G is centralizer connected with finite center, b then $Z(G) = \zeta_2(G)$, i.e. $Z(G/Z(G)) = \{1\}$.

Corollary 1.29. If G is centralizer connected, infinite and nilpotent, then $Z(G)$ is infinite.

Proof of Proposition 1.28. Recall that $\zeta_2(G) = \{g \in G | gZ(G) \in Z(G/Z(G))\}$. So for all $g \in \zeta_2(G)$, $h \in G$ we have $[g,h] \in Z(G)$.

Since $Z(G)$ is finite, we get for $g \in \zeta_2(G)$ that the orbit g^G is finite, so $[G : \mathrm{Cen}(g)]$ is finite. Hence $g \in Z(G)$, since G is centralizer connected. ☐

Remark 1.29.8. If G is nilpotent, $1 \neq N \trianglelefteq G$, then $N \cap Z(G) \neq \{1\}$:

Suppose $n \in (N \cap \zeta_i(G)) \setminus \{1\}$ with i minimal. If $i > 1$, then there exists $g \in G$ such that $1 \neq [g,n] \in \zeta_{i-1}(G) \cap N$.

Lemma 1.30. If G is nilpotent, centralizer connected and $N \trianglelefteq G$ infinite[a], then $N \cap Z(G)$ is infinite.

[a] not necessarily definable

Proof. If $N \leqslant Z(G)$ this is trivial. Otherwise $N \cap Z(G) \neq \{1\}$. If $1 \neq n \in N \cap \zeta_2(G) \setminus Z(G)$, then n^G is infinite and $n^{-1} \cdot n^G = [n, G] \subseteq Z(G) \cap N$ is infinite. ☐

Remark 1.30.9. If G is nilpotent, then for any subgroup $H \lneqq G$ we have $H \lneqq N_G(H)$ (cf. Sheet 1, Exercise 1 (B.1.1)).

Theorem 1.31. If G is stable, nilpotent, and $H < G$ definable of infinite index, then H has infinite index in $N_G(H)$.

Proof. Let $Z := Z(G)$. If $[ZH : H]$ is infinite, the claim is clear.

Now we use induction on the length of the central series: If $[ZH : H]$ is finite, then $[G : ZH] = [G/Z : ZH/Z]$ is infinite. By the inductive assumption ZH/Z

has infinite index in $N_{G/Z}(ZH/Z)$, hence ZH has infinite index in $N_G(ZH)$. We have

$$H \leqslant ZH \leqslant N_G(H) \leqslant N_G(ZH) =: N.$$

By Baldwin-Saxl (1.18) it is

$$\bigcap_{n\in\mathbb{N}} H^n = H^{n_1} \cap \ldots \cap H^{n_2} =: H^0$$

for some $l \in \mathbb{N}$.

Since $[ZH : H]$ is finite, H^0 has finite index in H and $N_G(ZH) \leqslant N_G(H^0)$. We obtain

$$H^0 \leqslant H \leqslant ZH,$$

where each step is of finite index. Hence

$$\left({}^{H}\!\!\diagup_{H^0}\right)^N \subseteq {}^{ZH}\!\!\diagup_{H^0}$$

is finite. Therefore $N_N(H)$ has finite index in N. Since $N_N(H) \leqslant N_G(H) \leqslant N$, the claim follows. □

[Lecture 04, 2024-04-25]

Remark 1.31.10. Note that when taking a quotient by a $\varnothing$-definable subgroup, e.g. $G/Z(G)$ in the proof of Theorem 1.31, the elements of the quotient are not elements of our structure. However the quotient is **interpretable** in G, i.e. equality up $Z(G)$ can be written as a formula in our language. We call elements of such an interpretable structure **virtual elements**.

More generally if E is a $\varnothing$-definable equivalence relation on M^n for some $\mathcal{L}$-structure M, $n \in \mathbb{N}$, we can extend the structure by a new **sort** of elements, whose elements are the equivalence classes modulo E. We extend the language $\mathcal{L}$ to a language $\mathcal{L}^{\mathrm{eq}}$ by adding for each such equivalence relation E a new sort and a new n-ary function symbol $\pi_E \colon M^n \to {}^{M^n}\!\!\diagup_E$.

Lemma 1.32. For every $\mathcal{L}^{\mathrm{eq}}$-formula $\varphi(x_1, \ldots, x_n)$, where x_1 is of the sort N^{n_i}/E_i, there is an $\mathcal{L}$-formula $\psi(\overline{y_1}, \ldots, \overline{y_n})$ which in T^{eq} is equivalent to $\varphi(\pi_{E_1}(\overline{y}_1), \ldots, \pi_{E_n}(\overline{y}_n))$.

Corollary 1.33. In M^{eq} there are no new definable relations on M. In particular, if M is stable / totally transcendental / NIP / ω-categorical then so is M^{eq}.

Example 1.34. If $H < G$ is 0-definable subgroup, then the cosets in G/H are the elements of the sort corresponding to $aE_Hb \iff ab^{-1} \in H$.

Furthermore if $H \trianglelefteq G$ is a normal subgroup then G/H is an interpretable group in G and is stable etc. if G is.

2 ω-categorical groups

Definition 2.1. A countable $\mathcal{L}$-structure M is called **ω-categorical** iff $\mathrm{Aut}(M)$ has only finitely many orbits on M^n for each n.

Example 2.2.

- $(\mathbb{Q}, <)$ is ω-categorical:

  Take $a_1 < \ldots < a_n$, and $b_1 < \ldots < b_n$, $a_i, b_i \in \mathbb{Q}$. Put $\varphi(a_i) := b_i$. Since $\mathbb{Q}$ is dense, φ can be extended to an automorphism of $\mathbb{Q}$.

- The random graph is ω-categorical.

- Vector spaces over a finite field K viewed as $(V, +, 0, \lambda_k : k \in K)$, where λ_k denotes scalar multiplication by k.

  Note that for an infinite field two elements can be linearly dependent in infinitely many ways. Hence vector spaces of an infinite field are not ω-categorical.

Remark 2.2.11.

(i) M is ω-categorical iff there is a unique countable structure elementarily equivalent[a] to M (up to isomorphism).

(ii) M is ω-categorical iff for any finite set $A \subseteq M$, $\mathrm{Aut}_A(M)$[b] has only finitely many orbits.

(iii) If M is ω-categorical and $A \subseteq M^n$ is invariant under $\mathrm{Aut}_B(M)$ for some finite set $B \overset{\text{finite}}{\subseteq} M$, then A is B-definable.

In particular if G is ω-categorical, then all characteristic subgroups are $\varnothing$-definable.

Exercise

Exercise

[a] $\mathcal{L}$-structures M, N are **elem**… the same sentences, i.e. $\{\varphi | M \models$ …

… notes the **point** …

Definition 2.3. A group G is called **locally finite** iff every finite subset generates a finite subgroup.

It is called **uniformly locally finite** iff for all $n \in \mathbb{N}$, there is a bound $k \in \mathbb{N}$, such that for all $a_1, \ldots, a_n \in G$, we have $|\langle a_1, \ldots, a_n \rangle| \leqslant k$.

In particular, a (uniformly) locally finite group is torsion (of bounded exponent).

Lemma 2.4. If G is an ω-categorical group, then G is uniformly locally finite.

Proof. Any automorphism of G fixing $a_1, \ldots, a_n$ fixes $\langle a_1, \ldots, a_n \rangle$ pointwise, hence $\langle a_1, \ldots, a_n \rangle$ is finite, as otherwise $\mathrm{Aut}_{a_1,\ldots,a_n}(G)$ has infinitely many orbits on M, one for each $x \in \langle a_1, \ldots, a_n \rangle$ (cf. Remark 2.2.11).

Since there are only finitely many orbits on n-tuples, and n-tuples in the same orbit generate isomorphic subgroups, the maximal bound works for all n-tuples. □

So far we have not used stability; now we'll add this assumption.

Theorem 2.5. If G is ω-categorical and stable, then the **connected component**

$$G^0 := \bigcap \{H < G | H \text{ definable (with parameters) of finite index}\}$$

is $\varnothing$-definable and of finite index.

Proof. If $H < G$ is definable (with parameters) and of finite index, then $H^0 := \bigcap_{\varphi \in \mathrm{Aut}(G)} \varphi(H)$ is a finite intersection (by Baldwin-Saxl (1.18)) and hence of finite index in G.

Since H^0 is a characteristic subgroup, it is $\varnothing$-definable. There are only finitely many such subgroups (cf. Remark 2.5.12 (i)), hence G^0 is $\varnothing$-definable and of finite index. □

Remark 2.5.12. (i) An ω-categorical group has only finitely many characteristic subgroups:

If $H \lhd_{\mathrm{char}} G$, $\tilde{G} := \mathrm{Aut}(G)$, then $x^{\tilde{G}} \in H$ or $x^G \cap H = \varnothing$ for all $x \in G$. Since there are only finitely many 1-orbits, the claim follows.

(ii) An ω-categorical stable group G contains minimal normal subgroups and any normal subgroup contains a minimal one:

There are only finitely many $\mathrm{Aut}(G)$-orbits on $G \times G$. Hence there is some $k \in \mathbb{N}$ such that for $x \in \langle y \rangle^G$ we have $x = y^{g_1} \cdot y^{g_i}$ for some $i \leqslant k$. Hence all normal subgroups of the form $\langle a^G \rangle$ are uniformly definable,

$$\langle a^G \rangle = \{a^{g_1} \cdot \ldots \cdot a^{g_i} | g_i \in G, i \leqslant k\}.$$

By the Trivial Chain Condition (1.13.4), there is a minimal one.

(iii) A stable group does not contain subgroups which are unbounded direct products of non-abelian groups.

If $H_1 \times \ldots \times H_k \leqslant G$, $h_i \in H_i \backslash Z(H_i)$, then $\bigcap_{j \neq i} \mathrm{Cen}(h_j) \geqslant H_i$ and $H_i \nleqslant \bigcap_{j \leqslant k} \mathrm{Cen}(h_j)$.

By the Corollary of Baldwin-Saxl (1.19), there is a bound on k depending only on $\mathrm{Th}(G)$.

(iv) Every finite simple group is 2-generated.

[Lecture 05, 2024-04-29]

Theorem 2.6 (Baw-Cherlin-Macintyre, Felgner)**.** An ω-categorical stable group G is virtually nilpotent.

TODO: Proof

[Lecture 06, 2024-05-02]

TODO

and 6)

3 Morley Rank

The **Morley rank** is a notion of dimension on definable sets, similarly to the algebraic dimension of an algebraic variety (and agrees with it in this context).

In this section let T always denote a complete theory with infinite models.

Definition 3.1. Let $\varphi(\overline{x})$ be an $\mathcal{L}(\mathcal{M})$-formula, $\mathcal{M} \models T$ very saturated.

(i) $\mathrm{MR}(\varphi) \geqslant 0$ if φ is consistent (i.e. $\varphi(M) \neq \varnothing$).

(ii) $\mathrm{MR}(\varphi) \geqslant \beta + 1$ if there is an infinite family of formulae $\varphi_i, i < \omega$ such that $\varphi_i(M) \cap \varphi_j(M) = \varnothing$ for $i \neq j$ and $\mathrm{MR}(\varphi_i) \geqslant \beta$ for all $i < \omega$.

(iii) $\mathrm{MR}(\varphi) \geqslant \lambda$ for limit ordinals λ if $\mathrm{MR}(\varphi) \geqslant \alpha$ for all $\alpha < \lambda$.

If φ is inconsistent, put $\mathrm{MR}(\varphi) = -\infty$. If $\mathrm{MR}(\varphi) \geqslant \alpha$ for all $\alpha \in \mathrm{Ord}$, put $\mathrm{MR}(\varphi) \coloneqq \infty$. If $\mathrm{MR}(\varphi) \geqslant \alpha$, $\mathrm{MR}(\varphi) \ngeqslant \alpha + 1$ put $\mathrm{MR}(\varphi) = \alpha$.

[Lecture 07, 2024-05-06]

Remark 3.1.13. (i) It is $\mathrm{MR}(\varphi) = 0$ iff $\varphi(\mathcal{M})$ is finite (in any model of T).

(ii) If for all $\mathcal{M} \models T$ we have $\varphi(\mathcal{M}) \subseteq \psi(\mathcal{M})$ (equivalently $T \vdash \varphi \to \psi$), then $\mathrm{MR}(\varphi) \leqslant \mathrm{MR}(\psi)$.

(iii) If $\mathrm{MR}(\varphi) = \alpha$ and $\beta \leqslant \alpha$ then there exists some ψ with $T \vdash \psi \to \varphi$ and $\mathrm{MR}(\psi) = \beta$.

Lemma 3.2. $\mathrm{MR}(\varphi \wedge \psi) = \max\{\mathrm{MR}(\varphi), \mathrm{MR}(\psi)\}$.

Proof. By Remark 3.1.13 (ii). $\mathrm{MR}(\varphi \wedge \psi) \geqslant \max\{\mathrm{MR}(\varphi), \mathrm{MR}(\psi)\}$. We show by induction on α, that if $\mathrm{MR}(\varphi \wedge \psi) \geqslant \alpha$, then $\max\{\mathrm{MR}(\varphi), \mathrm{MR}(\psi)\} \geqslant \alpha + 1$. If $\mathrm{MR}(\varphi \wedge \psi) \geqslant \alpha + 1$, then there exists $(\varphi_i)_{i<\omega}$, such that $T \vdash \varphi_i \to (\varphi \wedge \psi)$

and $\mathrm{MR}(\varphi_i) \geqslant \alpha$ for all $i < \omega$. By inductive assumption, $\mathrm{MR}(\varphi_i \wedge \varphi) \geqslant \alpha$ or $\mathrm{MR}(\varphi_i \wedge \psi) \geqslant \alpha$ for each $i < \omega$. Hence for φ or ψ there exists infinitely many i, such that $\mathrm{MR}(\varphi_i \wedge \varphi) \geqslant \alpha$ or $\mathrm{MR}(\varphi_i \wedge \psi) \geqslant \alpha$, so $\max\{\mathrm{MR}(\varphi), \mathrm{MR}(\psi)\} \geqslant \alpha + 1$. $\square$

Remark 3.2.14.

- If $\mathrm{MR}(\varphi) = \alpha$, then there exist only finitely many disjoint formulae $\varphi_1, \ldots, \varphi_d$ with $T \vdash \varphi_i \to \varphi$ and $\mathrm{MR}(\varphi_i) = \alpha$.

  The **Morley degree**, $\mathrm{Mdeg}(\varphi)$, is defined to be the maximum of all such d.

- φ, ψ are **disjoint** (over all models) if $T \cup \{\varphi, \psi\}$ is inconsistent.

Theorem 3.3. A theory T is totally transcendental iff every formula has a Morley rank.[a]

[a] i.e. $\mathrm{MR}(\varphi) \neq \infty$

Theorem 3.4. " $\Longrightarrow$ " Any formula without a Morley rank can be decomposed into an infinite binary tree.

" $\Longleftarrow$ " If $(\varphi_s)_{s \in {}^{<\omega}2}$ is a binary tree of consistent formulae, such that φ_s is of minimal Morley rank and Morley degree, then $\varphi_{s^\frown 0}$ and $\varphi_{s^\frown 1}$ have smaller Morley rank or Morley degree.

Definition 3.5. For types p we put

$$\begin{aligned} \mathrm{MR}(p) &:= \min\{\mathrm{MR}(\varphi) | \varphi \in p\}, \\ \mathrm{Mdeg}(p) &:= \min\{\mathrm{Mdeg}(\varphi) | \mathrm{MR}(\varphi) = \mathrm{MR}(p), \varphi \in p\}. \end{aligned}$$

Thus $\mathrm{MR}(\varphi) = \max\{\mathrm{MR}(p) | \varphi \in p\}$.

If G is a totally transcendental group, a formula $\varphi(x)$ and type $p(x)$ are called **generic** iff $\mathrm{MR}(\varphi) = \mathrm{MR}(p) = \mathrm{MR}(G) := \mathrm{MR}(\ulcorner x = x \urcorner)$.

We will need that in stable theories all types $p \in S(B)$, $B \subseteq \mathcal{M}$, $\mathcal{M} \models T$ are **definable**. First we do this for **φ-types**: We set $p \in S_\varphi(B)$ iff p is consistent and for every $\bar{b} \in B$ we have $\varphi(\bar{x}, \bar{b}) \in p$ or $\neg\varphi(\bar{x}, \bar{b}) \in p$.

Definition 3.6.

- A type $p \in S_n(B)$ is **definable** over C iff for each $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, there is an $\mathcal{L}(C)$-formula $\psi(y)$ such that for all $\bar{b} \in B$ we have $\varphi(\bar{x}, \bar{b}) \in p \iff \mathcal{M} \models \psi(\bar{b})$.
- $\varphi(\bar{x}, \bar{y})$ is called **stable** iff for some infinite cardinal λ we have $|S_\varphi(B)| \leqslant$

λ for all $|B| \leqslant \lambda$.

- $\varphi(\overline{x}, \overline{y})$ has the **order property (OP)** iff there are tuples $\overline{a_i}, \overline{b_i}$, $i < \omega$ such that $\mathcal{M} \models \varphi(a_i, b_j) \iff i < j$.
- $\varphi(x, y)$ has the **binary tree property** iff there is a binary tree $(b_s)_{s \in {}^{<\omega}2}$ of parameters such that for all $\sigma \in {}^{\omega}2$ the set

$$\{\varphi^{\sigma(n)}(\overline{x}, b_{\sigma|_n} | n < \omega\}$$

is consistent, where $\varphi^0 := \neg\varphi$ and $\varphi^1 := \varphi$.

Theorem 3.7. The following are equivalent:

(i) φ is stable.

(ii) $|S_\varphi(B)| \leqslant |B|$ for all infinite B.

(iii) φ doesn't have (OP).

(iv) φ doesn't have the binary tree property.

(v) Every φ-type $p \in S_\varphi(B)$ is definable over B.

For the proof we need some preparation:

Lemma 3.8. If $\varphi(\overline{x}, \overline{y})$ has (OP) and $(I; <)$ is a linear order, then there are a_i, b_i, $i \in I$ such that $\models \varphi(a_i, b_j)$ iff $i < j$.

Proof. Exercise. $\square$ TODO

Corollary 3.9. If $\varphi(\overline{x}, \overline{y})$ has (OP), then three are a_i, b_i, $i < \omega$ such that $\models \varphi(a_i, b_j)$ iff $i > j$.

We also need

Theorem 3.10 (Ramsey)**.** Let A be infinite, $n < \omega$, $C_1 \sqcup \ldots \sqcup C_k = [A]^n$ a colouring of the n-element subsets of A. Then there exists some infinite $A_0 \subseteq A$, $i \leqslant k$ such that $[A_0]^n \subseteq C_i$.

Proof. We use induction on n. The statement is trivial for $n = 1$. Assume that we have shown the theorem for some n. Consider a coloring c on $[A]^{n+1}$. Fix some $a_0 \in A$. We obtain a coloring on $[A \backslash \{a_0\}]^n$ as follows: For $[a_0] \cup X \in [A]^{n+1}$ put $c_{a_0}(X) := c(X \cup \{a_0\})$. By the induction hypothesis, there is a monochromatic set $B_1 \subseteq A \backslash \{a_0\}$. Take $a_1 \in B_1$. Color $[B_1 \backslash \{a_1\}]^n$ by $c_{a_1}(X) := c(X \cup \{a_0\})$. Iterating this construction we obtain a chain

$$A = B_0 \supseteq B_1 \supseteq B_1 \supseteq \ldots$$

and $a_i \in B_i \backslash B_{i+1}$ such that $C(\{a_{i_0}, a_{i_1}, \ldots, a_{i_n}\})$ depends only on i_0 for all $i_0 < i_1 < \ldots < i_n$. By induction hypothesis for $n = 1$, there are infinitely many i_0 yielding the same coloring. Let A_0 be the set of such i_0. ☐

Theorem 3.11 (Erdős-Makkai)**.** If B is infinite and $\mathcal{S} \subseteq \mathcal{P}(B)$ such that $|B| < |\mathcal{S}|$, then there is $\langle b_i | i < \omega \rangle$, $b_i \in B$, $\langle S_i | i < \omega \rangle$, $S_i \in \mathcal{S}$ such that either

(i) $b_i \in S_j \iff j < i$ or

(ii) $b_i \in S_j \iff i < j$.

Proof. We say that X *separates* A from B if $A \subseteq X$ and $X \cap B = \emptyset$. Construct $\mathcal{S}' \subseteq \mathcal{S}$, $|\mathcal{S}'| = |B|$ such that any pair of finite subsets of B that can be separated in $\mathcal{S}$ are separated in $\mathcal{S}'$: For any two finite subsets of B put a corresponding $B_0 \subseteq B$ into $\mathcal{S}'$.

Since $|\mathcal{P}_{\text{fin}}(B)| = |B|$, we have $|\mathcal{S}'| = |B|$. Since $|\mathcal{S}'| < |\mathcal{S}|$ there is $S^* \in \mathcal{S}$ which is *not* a boolean combination of sets in $\mathcal{S}'$. We now construct sequences

$$\begin{aligned} &\langle b_i' | i < \omega \rangle \text{ in } S^*, \\ &\langle b_i'' | i < \omega \rangle \text{ in } B \backslash S^*, \\ &\langle S_i | i < \omega \rangle \text{ in } \mathcal{S}', \end{aligned}$$

such that

- $\{b_0', \ldots, b_n'\} \subseteq S_n$, $\{b_0'', \ldots, b_n''\} \subseteq B \backslash S_n$ and
- $b_n' \in S_i \iff b_n'' \in S_i$ for all $i < n$.

Assume we have defined those for $i < n$. Since S^* is not a boolean combination of $S_i, i < n$, there exist $b_n' \in S^*$, $b_n'' \in B \backslash S^*$ such that for all $i < n$, $b_n' \in S_i \iff b_n'' \in S_i$. Let $S_n \in \mathcal{S}'$ separate $\{b_0', \ldots, b_n'\}$ from $\{b_0'', \ldots, b_n''\}$ (this exists, since $S^* \in \mathcal{S}$ separates them).

We may assume $b_n' \in S_i$ or $b_n' \notin S_i$ for all $i < n < \omega$: Set $c(\{n, m\}) := [b_{\max(n,m)} \in S_{\min(n,m)}]$. Ramsey's Theorem (3.10) yields $N \subseteq \omega$ infinite such that $[N]^2$ is monochromatic.

In the first case put $b_i := b_i''$ ($\notin S^*$, $\rightsquigarrow$ (i)). Otherwise put $b_i := b_{i+1}'$ ($\rightsquigarrow$ (ii)). By construction we have $i \leqslant n \implies b_i' \in S_n$, $b_i'' \notin S_n$. If $b_n' \in S_i$ for $i < n$, then also $b_n'' \in S_i$. Hence $i < n$ iff $b_n \in S_i$ by choice of S_n. The other case is similar. ☐

Proof of Theorem 3.7. Clearly (ii) $\implies$ (i), (v) $\implies$ (ii).

(i) $\implies$ (iv) Suppose that φ is λ-stable and μ minimal such that $2^\mu > \lambda$. The tree $T = {}^{<\mu}2$ has cardinality $\leqslant \lambda$. If $\varphi(\overline{x}, \overline{y})$ has the binary tree property, then by the Compactness Theorem (A.14) we find $(b_s)_{s\in T}$ such that for $\sigma \in {}^{\mu}2$ the type

$$q_\sigma := \{\varphi^{\sigma(x)}(\overline{x}, b_{\sigma|_\alpha} | \alpha < \mu\}$$

is consistent. Hence the q_σ extend to a family of pairwise distinct φ-types over $B = \{b_s | s \in T\}$, so $|B| \leqslant \lambda < 2^\mu \leqslant |S_\varphi(B)|$. ↯

(iv) $\implies$ (iii) Choose a linear ordering on $I = {}^{\leqslant\omega}2$ such that $\sigma < \sigma|_n \iff \sigma(n) = 1$ for all $\sigma \in {}^{\omega}2, n < \omega$. If $\varphi(x, y)$ has (OP), by Lemma 3.8 we find $(a_i, b_i)_{i\in I}$ such that $\models \varphi(a_i, b_j) \iff i < j$.

Thus the tree $\varphi(x, b_s)$, $s \in {}^{<\omega}2$ has the binary tree property.

(iii) $\implies$ (ii) Let $|B| \geqslant |T|$, $|S_\varphi(B)| > |B|$. The φ-type of a over B is determined by

$$S_a = \{\bar{b} \subseteq B | \models \varphi(a, \bar{b})\} \subseteq B^n.$$

Since $|B^n| = |B|$ we may assume $n = 1$. Applying Theorem 3.11 to B and $\mathcal{S} = \{S_a | a \in M\}$ we obtain $(b_i)_{i<\omega}$,$(a_i)_{i<\omega}$, $b_i \in B, a_i \in M$ such that either

- $b_i \in S_{a_j} \iff j < i$ for all $i, j < \omega$ or
- $b_i \in S_{a_j} \iff i < j$ for all $i, j < \omega$.

Thus φ has (OP).

(v) $\implies$ (iv) Suppose $\varphi(x, y)$ doesn't have the binary tree property. For a formula $\theta(x)$ let $d_\varphi(\theta)$ be the maximal n such that there is a binary three $(b_s)_{s\in {}^{<n}2}$ such that

$$\{\theta(x)\} \cup \{\varphi^{\sigma(i)}(x, b_{\sigma|_i} | i < n\}$$

is consistent for all $\sigma \in {}^{n}2$. Let $p \in S_\varphi(B)$ and let θ be a conjunction of formulae in p such that $n := d_\varphi(\theta)$ is minimal. Then

$$\varphi(x, b) \in p \iff d_\varphi(\theta(x) \wedge \varphi(x, b)) = n.$$

Note that the right hand side is definable. □

A Recap

[Tutorial 01, 2024-04-16]

A.1 Groups

Definition† A.0.15. Let G be a group. The **center** of G is

$$Z(G) := \{z \in G | \forall g \in G.\ zg = gz\}.$$

Definition† A.0.16. Let G be a group and $S \subseteq G$ a subset.

The **centralizer** of S in G is

$$C_G(S) := \{g \in G | \forall s \in S.\ gs = sg\}.$$

The **normalizer** of S in G is

$$N_G(S) := \{g \in G | gS = Sg\}.$$

Clearly $C_G(S) \leqslant N_G(S) \leqslant G$ are subgroups.

Let $A, B \leqslant G$ be two subgroups. Then A is **normalized** by B (B **normalizes** A) iff for every $b \in B$, $A^b = A$, i.e. $B \leqslant N_G(A)$.

Similarly, A is **centralized** by B (B **centralizes** A) iff $B \leqslant C_G(A)$.

Definition† A.0.17. Let p be prime. A ***p*-group** is a group in which the order of every element is a power of p.

Let G be a group. A ***p*-Sylow subgroup** of G is a maximal p-subgroup of G.

Sylow theorems

Definition† A.0.18. Let G be a group. The **order** of G, $\mathrm{ord}(G)$, is the number of elements of G.

The **order** (**period length** / **period**) of $g \in G$ is the order of $\langle g \rangle$, the subgroup generated by g.

G is a **torsion group** (**periodic group**) iff every element has finite order.

The **exponent** of a torsion group G is the least common multiple of orders of the group elements, i.e. the least $n \in \mathbb{N}$ such that $\forall g \in G.\ g^n = 1$. (This does not necessarily exist.)

Definition A.1. A group G is called **simple** iff $\{1\}$ and G itself are the only normal subgroups of G.

Semidirect product

A.1.1 Group actions

Definition A.2. Let G be a group and X a set. A **group action** $G \curvearrowright X$ is a group homomorphism $\pi: G \to \mathrm{Sym}(X)$. For $g \in G, x \in X$ we will write $g.x := \pi(g)(x)$.

- The action is **transitive** iff $\forall x, x' \in X.\ \exists g \in G.\ gx = x'$.
- The action is **faithful** iff π is surjective.
- The action is **free** iff no non-trivial element has a **fixpoint**, i.e. $\forall g \in G \backslash \{1\}, x \in X.\ g.x \neq x$.
- The action is **regular** iff it is transitive and free.
- The **stabilizer** of $x \in X$ is the subgroup $G_x := \{g \in G : g.x = x\}$.
- The **orbit** of $x \in X$ is $G.x := \{g.x | g \in G\}$.

Definition A.3. For a subgroup $H \leqslant G$ the **index** of H in G, $|G : H|$ is defined as $|\{gH/g \in G\}|$.

Theorem A.4 (Orbit stabilizer theorem)**.** Let $G \curvearrowright X$, $x \in X$. Then $|G.x| = |G : G_x|$.

Proof. The map

$$\begin{aligned} \varphi: G.x &\longrightarrow G/G_x \\ g.x &\longmapsto gG_x \end{aligned}$$

is bijective. □

A.1.2 Nilpotent and solvable groups

Definition A.5. Let G be a group. The **commutator** or **derived subgroup** of G, denoted $[G, G]$ or G', is defined as $\{[g, g'] : g, g' \in G\}$. $[G, G]$ is the smallest subgroup of G such that G/G' is abelian.

We recursively define $G^{(0)} := G$ and $G^{(n+1)} := [G^{(i)}, G^{(i)}]$.

We say that G is **solvable** if $G^{(n)} = 1$ for some $n \in \mathbb{N}$.

Proposition A.6. G is solvable iff we have a sequence $1 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_m = G$ such that H_{i+1}/H_i is abelian.

Example A.7. Let K be a field, $\mathrm{AGL}_1(K) := \{x \mapsto ax+b | a \neq 0, b \in K\} \cong K \rtimes K$ the group of affine transformations.

We have $\mathrm{AGL}_1(K)/K^+ \cong K^+$, so $1 \trianglelefteq K^+ \trianglelefteq \mathrm{AGL}_1(K)$.

Definition A.8. A **linear algebraic group** of a field k is a subgroup of $\mathrm{GL}_1(K)$, e.g. $\mathrm{SL}_1(K)$. For such a group, a Borel subgroup is a maximal closed connected solvable group.

Example A.9.

- A Borel subgroup for $\mathrm{GL}_n(K)$ is the group of upper triangular matrices.
- A Borel subgroup for $\mathrm{SL}_n(K)$ is the group of upper **unitriangular** matrices (i.e. upper triangular matrices with only 1 on the diagonal).

Definition A.10. Let G be a group. We define $G^{[n]}$ inductively be $G^{[0]} := G$, $G^{[n]} := [G, G^{[n-1]}]$.

We say that G is (n-step) **nilpotent** iff there exists $n \in \mathbb{N}$ (minimal) such that $G^{[n]} = 1$.

Nilpotent groups are solvable.

Proposition A.11. The following are equivalent

- G is nilpotent.
- There exists a finite ascending central series, $\zeta_0(G) = 1 \trianglelefteq \zeta_1(G) = Z(G) \trianglelefteq \ldots \trianglelefteq \zeta_n(G) = G$ with $\zeta_{i+1}(G) = \{g \in G : g\zeta_i(G) \in Z(G/\zeta_i(G))\}$, that is $\zeta_{i+1}(G)$ is the subgroup such that $\zeta_{i+1}(G)/\zeta_i(G) = Z(G/\zeta_i(G))$.

Example A.12.

- For a field K, the group of upper unitriangular matrices is nilpotent.
- The **quaternion group** $Q_8 = \langle a, b | a^4 = e, a^2 = b^2, ba = a^{-1}b \rangle$ is nilpotent.

If a group is nilpotent / solvable, then its subgroups have the same property.

A.2 Model Theory

It was recommended to read the first two chapters of Prof. Tent's book "A Course in Model Theory" (Structures, Languages, Theories, Elementary Substructures, Compactness Theorem, Löwenheim-Skolem) as an introduction to model theory.

A.2.1 The Compactness Theorem

Definition A.13. Let $\mathcal{L}$ be a first order languages and Σ a set of sentences in the language $\mathcal{L}$. We say that Σ is **satisfiable** iff there exists an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \sigma$ for all $\sigma \in \Sigma$ (or short $\mathcal{M} \models \Sigma$).

We say that Σ is **finitely satisfiable** iff every finite subset $\Sigma' \subseteq \Sigma$ is satisfiable.

Theorem A.14 (Compactness Theorem)**.** Σ is satisfiable iff it is finitely satisfiable.

B Exercises

B.1 Sheet 1

[Tutorial 02, 2024-04-23]

Definition B.1. The **ascending central series** of a group G is the chain $Z_0(G) \leqslant Z_1(G) \leqslant \ldots$ of subgroups of G defined inductively by $Z_0 := \{1\}$ and $Z_i(G)/Z_{i-1}(G) := Z(G/Z_{i-1}(G))$ for $i > 0$. A group G is **nilpotent** if $Z_n(G) = G$ for some $n \in \mathbb{N}$.

Definition B.2. A **Sylow-p-subgroup** of G is a p-group that is maximal wrt. inclusion.

Usually this is only defined for finite groups, where Sylow's theorem holds. However, we also consider Sylow-subgroups of infinite groups. The following version of Sylow's theorem holds:

Theorem B.3 (Sylow's theorem for infinite groups)**.** Let P be a Sylow-p-subgroup of G. If P has finitely many conjugacy classes in G, then all Sylow-p-subgroups of G are conjugate.

Definition B.4. A **characteristic subgroup** is a subgroup that is fixed by every automorphism.

This is a stronger condition than being normal (i.e. fixed under the automorphisms of the for $h \mapsto ghg^{-1}$.)

B.1.1 Exercise 1

TODO

B.1.2 Exercise 2

Let G be a group and $P \leqslant G$ be a p-Sylow subgroup of G.

(a) Show that P is a characteristic subgroup of $N_G(P)$. Deduce that $N_G(N_G(P)) = N_G(P)$.

(b) Suppose that G is nilpotent.

- Show that $N_G(P) = G$, i.e. P is normal in G.
- Deduce that G has a unique p-Sylow subgroup for each prime p.
- Conclude that any finite nilpotent group is the direct sum of its p-Sylow subgroups.

B.1.3 Exercise 3

(a) Let G be a group of exponent p,[7] where p is prime. Let $g \in G\backslash\{1\}$.

Show that no two distinct elements of $\langle g \rangle$ are conjugate and deduce that G has at least p conjugacy classes.

(b) Let G be a group. Suppose $g \in G\backslash\{1\}$ has finite order and $G = g^G \cup \{1\}$. Show that $|G| = 2$.

B.1.4 Exercise 4

> **Definition B.5.** A transitive group action $G \curvearrowright X$, with $|X| > 1$ is called **primitive** iff each stabilizer G_x is a maximal proper subgroup of G.

(a) Let G be a group acting transitively on a set X.

Suppose that for some $x \in X$, the stabiliser G_x is normal. Show

(i) $G_y = G_x$ for every $y \in X$.

(ii) G/G_x acts regularly[8] on X.

(b) Let $G \curvearrowright X$ be primitive and G nilpotent. Then $|X|$ is prime.

B.2 Sheet 2

[Tutorial 03, 2024-04-30]

B.2.1 Exercise 1

Let G be a group considered as a structure in a language $\mathcal{L} \supseteq \mathcal{L}_{\text{group}}$.

(a) Let $\varphi(x, y_1, \ldots, y_s)$ be an $\mathcal{L}$-formula such that for any $\overline{g} \in G^s$, $H_{\overline{g}} := \varphi(G, \overline{g})$ is a finite index subgroup of G.

Then the following are equivalent:

(i) The index is uniformly bounded, i.e. $\exists n \in \mathbb{N}.\ \forall \overline{g} \in G^s.\ |G : H_{\overline{g}}| < n$.

[7] We say that G is a group of **exponent** p iff $g^p = 1$ for all $g \in G$.
[8] A group action is called **regular** iff $g \mapsto gx$ is bijective for all x.

(ii) For every model G' of $\mathrm{Th}(G)$ and every $\overline{g} \in (G')^s$, $H_{\overline{g}} = \varphi(G', \overline{g})$ is a finite index subgroup of G'.

(b) Suppose that for every $G' \equiv G$ and $g \in G'$, the conjugacy class $g^{G'}$ is finite. Then $|g^{G'}|$ is uniformly bounded.

B.2.2 Exercise 2

Let G be an infinite stable group. Show that $G \backslash \{1\}$ does not form a single conjugacy class.

B.2.3 Exercise 3

Let G be an NIP group in a language $\mathcal{L}$. Let $\varphi(x, y_1, \ldots, y_s)$ be an $\mathcal{L}$-formula such that $H_{\overline{g}} := \varphi(G, \overline{g})$ is a subgroup for every $\overline{g} \in G^s$. Let $k \in \mathbb{N}$ and let

$$X_k := \{\overline{g} \in G^s : |G : H_{\overline{g}}| \leqslant k\} \subseteq G^s.$$

Show that $\bigcap_{\overline{g} \in X_k} H_{\overline{g}}$ is a finite index subgroup of G which is $\varnothing$-definable.

B.2.4 Exercise 4

Let G be a stable group. Then any abelian subgroup of G is contained in a definable abelian subgroup of G.

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