Stable Groups

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These are my notes on the lecture "Stable Groups", taught by Prof. Dr. Dr. Katrin Tent in the summer term 2024 at the University Münster.

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[Lecture 01, 2024-04-15]

A background in model theory is helpful but not necessary. Some group theory is required (usually covered in linear algebra and a first algebra course).

The lecture starts at 08:25.

The book by Prof. Tent is available on learnweb.

There will be an oral exam. For a type II course, one needs to do nothing.

The main point is to see, how model theoretic properties influence algebraic properties.

1 Introduction

Definition 1.1. An infinite $\mathcal{L}$-structure M is **minimal**, iff for every formula $\varphi(x) \in \mathcal{L}(M)$, the set defined by φ, $\varphi(M) := \{a \in M | M \models \varphi(a)\}$ is finite or cofinite.

Example 1.2. $(\mathbb{Q}, +, \cdot)$ is not minimal, consider or example the formula $\varphi(x) := \exists y.\ x = y^2$

$(\mathbb{C}, +, \cdot)$ is minimal.

Recall the **orbit equation**: If $G \curvearrowright X$ is transitive, then there is a natural bijection

$$\begin{aligned} G/G_x &\longrightarrow X \\ hG_x &\longmapsto h \cdot x \end{aligned}$$

where for $x \in X$, $G_x := \{g \in G : gx = x\} \leqslant G$ is the **stabilizer** of x in G and $G \cdot x := \{gx : g \in G\} \subseteq X$ is the **orbit** of x under G.

Theorem 1.3 (Reineke)**.** Minimal groups are abelian.

Proof. Let G be a minimal group.

Since G is minimal, all proper definable[1] subgroups are finite by minimality: If $H \lneq G$ is a proper definable subgroup, then for $a \notin H$, the coset $a \cdot H$ is also definable and disjoint from H.

Suppose that G is not abelian. Then the center[2] $Z(G)$ is finite. Furthermore, every element of the group must have finite order, since $\langle a \rangle \leqslant Z(\mathrm{Cen}(a))$.[3] (Note that $\langle a \rangle$ is not definable in general).

[1] A **definable** subgroup is a subgroup, that can be defined by a formula.
[2] The **center** is defined as $Z(G) := \{x \in G : \forall y.\ xy = yx\}$.
[3] The **centralizer** of a is the set of all elements commuting with a.

Consider the conjugacy class $a^G := \{a^g : g \in G\}$, where $a^g := g^{-1}ag$. Then for $a \in Z(G)$,[4] we have $|a^G| = |{}^G\!/\!_{G_x}|$, where $G_a = \mathrm{Cen}(a)$. In particular, for $a \notin Z(G)$, the conjugacy class a^G is infinite. Since by minimality there can not be two disjoint infinite conjugacy classes, we get $G = a^G \cup Z(G)$ for all $a \notin Z(G)$. Thus any $a, b \in G \backslash Z(G)$ are conjugate, so a, b have the same finite order and $|\mathrm{Cen}(a)| = |\mathrm{Cen}(b)|$.

If all elements have order 2, the group is abelian, since $a^{-1}b^{-1}ab = abab = 1$ in this case.

If all $a \in G \backslash Z(G)$ have order 2, then again G is abelian: Let $c \in Z(G)$, then $ca \notin Z(G)$, so $1 = (ac)^2 = acac = a^2c^2 = c^2$, i.e. the elements in $Z(G)$ also have order 2.

Now let $a \in G \backslash Z(G)$. Then $a^2 \neq 1$ and $a, a^{-1} \notin Z(G)$ are conjugate under some $g \in G$, i.e. $b^{-1}ab = a^{-1}$, hence $b^{-2}ab^2 = a$, $b^2 \in \mathrm{Cen}(a)$. So $a \in \mathrm{Cen}(b^2) \backslash \mathrm{Cen}(b)$. Clearly $\mathrm{Cen}(b) \leqslant \mathrm{Cen}(b^2)$ and a witnesses that this is a proper subgroup. So $|\mathrm{Cen}(b)| \neq |\mathrm{Cen}(b^2)|$, hence $b^2 \in Z(G)$. It follows that $H = {}^G\!/\!_{Z(G)}$ is an elementary abelian 2-group in which all non-trivial elements are conjugate, i.e. $|H| = 2$[5] and so G is finite. □

We want to generalize this.

Definition 1.4. An $\mathcal{L}$-structure M is **stable** iff there are no $M \preceq \tilde{M}$,[a] $\mathcal{L}(\tilde{M})$-formula $\varphi(\overline{x}, \overline{y})$ and tuples $\overline{a_i}, \overline{b_j} \in \tilde{M}$ such that $\tilde{M} \models \varphi(\overline{a_i}, \overline{b_j})$ iff $i < j$.

[a] elementary extension

Example 1.5. Let $M = (\mathbb{Z}, +, \cdot, 0, 1)$, $a_i = i = b_i$ and $\varphi(x, y) \longleftrightarrow \ulcorner \exists z_1, \ldots, z_4.\ x + z_1^2 + \ldots + z_4^2 = y \urcorner$. Then $M \models \varphi(a_, b_j)$ iff $i \leqslant j$. So M is not stable.

Algebraically closed fields are stable.

Lemma 1.6. If M is a stable and non-empty **semigroup**[a] with right- and **left-cancellation**[b] (alternatively: left-cancellation and a **right neutral element**[c]), then M is abelian.

[a] associative operation
[b] $ax = ay \implies x = y$
[c] $\forall a.\ ae = a.$

Proof. The formula $\varphi(x, y) \longleftrightarrow \ulcorner \exists z.\ x \cdot z = y \urcorner$, is satisfied by (a^n, a^m) if $n < m$. By stability, this can not be an if and only if. So there must be some $m > n$,

[4] Note that for $a \in Z(G)$, we have $a^G = \{a\}$.
[5] Conjugation is not too interesting in abelian groups.

such that $M \models \varphi(a^n, a^m)$. I.e. there is some $b \in M$ such that $a^n = a^{n+p}b$, where $m = n + p$. Put $e = a^p b$. This is a left-neutral element: For $c \in M$ we have $a^n c = a^n ec$, hence $c = ec$ by left-cancellation.

By symmetry (or assumption), there exists a right-neutral element f, and since $e = ef = f$, e is neutral.

Furthermore

$$e = a^p b \overset{p \geq 0}{=} a(a^{p-1}b),$$

so a has an inverse. □

Remark 1.6.1. The assumptions are necessary since a semigroup with $xy = y$ is not a group.

Corollary 1.7. If G is stable, then every non-empty definable subset closed under multiplication is a subgroup.

Similarly, every definable non-empty subring of a stable field is a subfield.

Remark 1.7.2. A stable group is a group whose theory is stable, (not necessarily in the language of groups). The group may be a (or interpretable) structure inside another structure, e.g. $(K, +, \cdot, 0, 1)$ field, $G = \mathrm{GL}_n(K)$ or any other Chevalley group.

Definition 1.8. A definable group action (in some L-structure M) is given by a definable group G, a definable set X and a definable action $G \times X \to X$ (i.e. the graph of the action is a definable subset of $(G \times X) \times X$.

Example 1.9. Let $(K, +, \cdot, 0, 1)$ be a field. Then $\mathrm{GL}_n(K)$, K^n and $\mathrm{GL}_n(K) \curvearrowright K^n$ are definable.

Example 1.10. Consider $(\mathbb{Q}, +, \cdot, 0, 1)$. Then $A := [0, 1]$ is definable[a] and $\frac{1}{n}A \subsetneq A$. Hence it is not stable by the following lemma.

[a] this is non-trivial

Lemma 1.11. Let G be a stable group acting definably on a set X. If $A \subseteq X$ is definable and $g \in G$, then $g(A) \subseteq A$ iff $g(A) = A$.

Proof. If $g(A) \subsetneq A$, we get a proper descending sequence $A \supsetneq g(A) \supsetneq g^2(A) \supsetneq g^3(A) \supsetneq \dots$ and the sequence g^i, $i < \omega$ is ordered by $\ulcorner xA \subsetneq yA \urcorner$. □

[Lecture 02, 2024-04-18]

Recall:

Corollary 1.12 (of Lemma 1.11)**.** If G is stable, $A \subseteq G$ is definable and $g \in G$, then $A^g \leqslant A \iff A^g = A$.

Remark 1.12.3. This does not hold in general. Consider

$$H = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\} < \mathrm{GL}_2(\mathbb{Q})$$

and

$$g := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, h_m := \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}.$$

Then $gh_m g^{-1} = h_{2m}$, so $gHg^{-1} \lneq H$.

Definition 1.13. For a group G, a family $\{H_i\}_{i \in I}$ of subsets $\{H_i\}_{i \in I}$ of M^k is called **uniformly definable** if there is a formula $\varphi(\overline{x}, \overline{y})$ and $\overline{a_i} \in M_i, i \in I$ such that $\varphi(M^k, \overline{a_i}) = H_i$.

For example, the centralizers of elements are uniformly definable.

Remark 1.13.4. If G is stable, the **Trivial Chain Condition** holds for uniformly definable subsets and subgroups, i.e. descending chains are finite:

For every uniformly definable family $H_i, i \in I$, there is some $n < \omega$ such that every properly descending (resp. ascending) chain $H_{i_1} \lneq H_{i_2} \lneq H_{i_3} \lneq \ldots$ has length at most n. This n depends only on the formula, not on the parameters of the form of the definable family.

Definition 1.14. A formula $\varphi(\overline{x}, \overline{y})$ has the **independence property** (**IP**) iff there are $\overline{a_i}, i < \omega$, such that for all $A \subseteq \omega$, the set $\{\varphi(\overline{x}, \overline{a_i}) | i \in A\} \cup \{\neg\varphi(\overline{x}, \overline{a_i}) | i \notin A\}$ is consistent.

relate to wikipedia definition (use compactness), see proof of Lemma 1.17

A theory is called **NIP** iff no consistent formula has IP.

Example 1.15. The random graph (Radograph) has IP. $(\mathbb{C}, +, \cdot, 0, 1)$ is NIP.

Lemma 1.16. If T is stable, then T is NIP.

Proof. If $\varphi(\overline{x}, \overline{y})$ has IP, $\overline{a_i} \in M, i < \omega$, $M \models T$, then there are $\tilde{M} \succeq M$, and $b_i \in \tilde{M}$ such that $\tilde{M} \models \varphi(\overline{b_i}, \overline{a_j})$ iff $i < j$, which is a contradiction to stability. $\square$

The reverse direction does not hold, since for example the real numbers have

NIP, but are not stable.

Lemma 1.17. Let G be a NIP group. Then finite intersections of uniformly definable subgroups are uniformly bounded, i.e. for every formula $\varphi(x,\overline{y})$ there is $n<\omega$ such that if $H_i=\varphi(G,\overline{a_i})$, $i=1,\dots,m$ are subgroups, then

$$\bigcap_{i\leqslant m} H_i=\bigcap_{j=1}^{n} H_{i_j}.$$

Proof. Suppose not. Then for all $n<\omega$ there is a uniformly definable family of subgroups $H_1,\dots,H_n$ such that $\bigcap H_i \lneq \bigcap_{\substack{i=1\\ i\neq j}}^{n} H_i$ for any $1\leqslant j\leqslant n$.

So there is some $b_j\in\left(\bigcap_{i\neq j} H_i\right)\backslash H_j$, $j\leqslant n$.

For $I\subseteq\{1,\dots,n\}$ put $b_I:=\prod_{i\in I} b_i$. Then $G\models\varphi(b_j,\overline{a_i})$ iff $i\notin J$. Since n was arbitrary, this shows that $\varphi(x,y)$ has IP: Let $A\subseteq\omega$ be any subset. By the compactness theorem it suffices to show that every finite subset of $\{\varphi(x,\overline{a_i})|i\in A\}\cup\{\neg\varphi(x,\overline{a_i})|i\notin A\}$ is consistent. This holds, since for every $I\overset{\text{finite}}{\subseteq}\omega$,

$$G\models\{\varphi(b_{I\backslash A},\overline{a_i})|i\in A\cap I\}\cup\{\neg\varphi(b_{I\backslash A},\overline{a_i})|i\in I\backslash A\}.$$

□

Proposition 1.18 (Baldwin-Saxl)**.** If G is stable, then for every formula $\varphi(\overline{x},\overline{y})$, there is $n<\omega$ (depending only on φ) such that for subgroups $H_i=\varphi(G,\overline{a_i})_{i\in I}$, we have

$$\bigcap_{i\in I} H_i=\bigcap_{j=1}^{n} H_{i_j}$$

for some $i_j\in I$, i.e. arbitrary intersections of uniformly definable subgroups are definable.

Proof. By Lemma 1.17 intersections of finitely many H_i are uniformly definable. By the Trivial Chain Condition (1.13.4) applied to these uniformly definable intersections, there is a minimal group H is this family, i.e. $H=\bigcap H_i=\bigcap_{j=1}^{n} H_{i_j}$ with $n<\omega$ from Lemma 1.17. □

Corollary 1.19. If G is stable and $A\leqslant G$ arbitrary, then $\mathrm{Cen}(A)=\bigcap_{a\in A}\mathrm{Cen}(a)=\{g\in G|\forall a\in A.\ [g,a]=1\}$ is definable.

Example 1.20. By Sela's Theorem the free groups F_n are stable. For $w \in F_k$, $\mathrm{Cen}(w)$ is cyclic, so $n = 2$.

Copy from oficial notes

Remark 1.20.5. Since the formula $\ulcorner xa = ax \urcorner$ is quantifier-free, Corollary of Baldwin-Saxl (1.19) holds in all subgroups of stable groups.

For example $\mathrm{Sym}_{\mathrm{fin}}(\omega)$, the group of permutations of ω with finite support (i.e. moving only finitely many elements) can never be a subgroup of a stable group, since centralizers can become arbitrarily small.

Definition 1.21. Let T be arbitrary and $\varphi_s(\overline{x}), s \in 2^{<\omega}$ consistent formulae.[a]

Then

(i) the $\varphi_s(\overline{x})$ form a **binary tree of consistent formulae** iff

$$T \vdash \forall \overline{x}(\varphi_{s^\frown 0}(\overline{x}) \vee \varphi_{s^\frown 1}(\overline{x}) \rightarrow \varphi_s(\overline{x}))$$

and

$$T \vdash \forall \overline{x} \neg(\varphi_{s^\frown 0}(\overline{x}) \wedge \varphi_{s^\frown 1}(\overline{x})).$$

(ii) T is called **totally transcendental** (or **ω-stable** iff $\mathcal{L}$ is countable) iff there is no binary tree of consistent formulae.

[a] Here "consistent" means that the family is consistent along every path, i.e. for every $s \in 2^\omega$, $\{\varphi_{s|_n} : n \in \omega\}$ is consistent. The entire family may be inconsistent.

Example 1.22. Let G be a group, and $H_i, i < \omega$ an infinite descending chain of subgroups $H_i \gneq H_{i+1}$, then we get a binary tree (subset vs. coset). So totally transcendental is much stronger than stable.

Proposition 1.23. If G is totally transcendental, there is no infinite properly descending chain of definable subgroups.

Proof. Otherwise we get a binary tree.

□

[Lecture 03, 2024-04-22]

Corollary 1.24.

(i) In a totally transcendental group G every intersection of definable subgroups is definable. In particular, there is a minimal definable subgroup G^0 of finite index in G, the **connected component** of G.

(ii) If G is totally transcendental, every injective definable endomorphism of G is surjective, i.e. an automorphism of G.

(iii) If G is ω-stable, abelian and torsion free, then G is divisible[a].

[a] An abelian group A is **divisible** iff $\forall a \in A.\ \forall n \in \mathbb{N}.\ \exists b \in A.\ n \cdot b = a$, i.e. iff $G \cong \bigotimes_{i\in I}\mathbb{Q}$.

Proof. (i) Clear.

(ii) Suppose that $s\colon G \hookrightarrow G$ is definable but not surjective. Then $s^i(G)$ is a proper descending sequence of definable subgroups $\lightning$.

(iii) Note that the map $g \mapsto n \cdot g$ is definable and injective.[6]

□

Remark 1.24.6. If G is stable, then for any formula $\varphi(x, \overline{y})$ the group

$$G^0(\varphi) = \bigcap\{\varphi(G, \overline{a}) | \varphi(G, \overline{a}) \leqslant G, [G : \varphi(G, \overline{a})] < \infty\}$$

is a definable subgroup of finite index by Baldwin-Saxl (1.18), the **φ-connected component of G**.

In particular, we'll be interested in the case

$$\varphi(x, y) \longleftarrow \ulcorner xy = yx \urcorner.$$

Definition 1.25. A group G is called **centralizer connected** iff $G = G^0(xy = yx)$, i.e. iff for all $a \in G\backslash Z(G)$ the index $[G : \mathrm{Cen}(a)]$ is infinite.

Lemma 1.26. If G is centralizer connected, $A \subseteq G$ finite and A normalized by[a] G, then $A \subseteq Z(G)$.

[a] For $A, B \leqslant G$ we say that A is **normalized** by B iff $\forall b \in B.\ A^b = A$, i.e. $B \leqslant N_G(A)$.

Proof. If a^G is finite, then $a \in Z(G)$, since $|G : \mathrm{Cen}(a)| = |a^G|$. □

Remark 1.26.7. This does not depend on stability.

Proposition 1.27. If G is stable[a] and $\{[g, h] | g, h \in G\}$ finite, then G is virtually abelian.[b]

[a] The assumption of G being stable can be removed.

[6] Warning: $g \mapsto n \cdot g$ is not uniformly definable.

[b] A group is called **virtually abelian** or **abelian-by-finite** iff $Z(G)$ has finite index in G.

Proof. For every $g \in G$, the set $\{[g,h] : h \in G\}$ is finite. Hence g^G is finite, so $|G : \mathrm{Cen}(g)|$ is finite. By the Corollary of Baldwin-Saxl (1.19), we have $Z(G) = \bigcap_{i \leqslant n} \mathrm{Cen}(g_i)$ for some $n \in \mathbb{N}$, and this has finite index. ☐

Proposition 1.28. If G is centralizer connected with finite center, b then $Z(G) = \zeta_2(G)$, i.e. $Z(G/Z(G)) = \{1\}$.

Corollary 1.29. If G is centralizer connected, infinite and nilpotent, then $Z(G)$ is infinite.

Proof of Proposition 1.28. Recall that $\zeta_2(G) = \{g \in G | gZ(G) \in Z(G/Z(G))\}$. So for all $g \in \zeta_2(G)$, $h \in G$ we have $[g,h] \in Z(G)$.

Since $Z(G)$ is finite, we get for $g \in \zeta_2(G)$ that the orbit g^G is finite, so $[G : \mathrm{Cen}(g)]$ is finite. Hence $g \in Z(G)$, since G is centralizer connected. ☐

Remark 1.29.8. If G is nilpotent, $1 \neq N \trianglelefteq G$, then $N \cap Z(G) \neq \{1\}$:

Suppose $n \in (N \cap \zeta_i(G)) \backslash \{1\}$ with i minimal. If $i > 1$, then there exists $g \in G$ such that $1 \neq [g,n] \in \zeta_{i-1}(G) \cap N$.

Lemma 1.30. If G is nilpotent, centralizer connected and $N \trianglelefteq G$ infinite[a], then $N \cap Z(G)$ is infinite.

[a] not necessarily definable

Proof. If $N \leqslant Z(G)$ this is trivial. Otherwise $N \cap Z(G) \neq \{1\}$. If $1 \neq n \in N \cap \zeta_2(G) \backslash Z(G)$, then n^G is infinite and $n^{-1} \cdot n^G = [n,G] \subseteq Z(G) \cap N$ is infinite. ☐

Remark 1.30.9. If G is nilpotent, then for any subgroup $H \lneq G$ we have $H \lneq N_G(H)$ (cf. **??**).

Theorem 1.31. If G is stable, nilpotent, and $H < G$ definable of infinite index, then H has infinite index in $N_G(H)$.

Proof. Let $Z := Z(G)$. If $[ZH : H]$ is infinite, the claim is clear.

Now we use induction on the length of the central series: If $[ZH : H]$ is finite, then $[G : ZH] = [G/Z : ZH/Z]$ is infinite. By the inductive assumption ${}^{ZH}\!/_{Z}$

has infinite index in $N_{G/Z}(ZH/Z)$, hence ZH has infinite index in $N_G(ZH)$. We have

$$H \leqslant ZH \leqslant N_G(H) \leqslant N_G(ZH) =: N.$$

By Baldwin-Saxl (1.18) it is

$$\bigcap_{n \in \mathbb{N}} H^n = H^{n_1} \cap \ldots \cap H^{n_2} =: H^0$$

for some $l \in \mathbb{N}$.

Since $[ZH : H]$ is finite, H^0 has finite index in H and $N_G(ZH) \leqslant N_G(H^0)$. We obtain

$$H^0 \leqslant H \leqslant ZH,$$

where each step is of finite index. Hence

$$\left({}^{H}\!/\!_{H^0}\right)^N \subseteq {}^{ZH}\!/\!_{H^0}$$

is finite. Therefore $N_N(H)$ has finite index in N. Since $N_N(H) \leqslant N_G(H) \leqslant N$, the claim follows. □

[Lecture 04, 2024-04-25]

Remark 1.31.10. Note that when taking a quotient by a $\varnothing$-definable subgroup, e.g. $G/Z(G)$ in the proof of Theorem 1.31, the elements of the quotient are not elements of our structure. However the quotient is **interpretable** in G, i.e. equality up $Z(G)$ can be written as a formula in our language. We call elements of such an interpretable structure **virtual elements**.

More generally if E is a $\varnothing$-definable equivalence relation on M^n for some $\mathcal{L}$-structure M, $n \in \mathbb{N}$, we can extend the structure by a new **sort** of elements, whose elements are the equivalence classes modulo E. We extend the language $\mathcal{L}$ to a language $\mathcal{L}^{\mathrm{eq}}$ by adding for each such equivalence relation E a new sort and a new n-ary function symbol $\pi_E \colon M^n \to {}^{M^n}\!/\!_{E}$.

Lemma 1.32. For every $\mathcal{L}^{\mathrm{eq}}$-formula $\varphi(x_1, \ldots, x_n)$, where x_1 is of the sort N^{n_i}/E_i, there is an $\mathcal{L}$-formula $\psi(\overline{y_1}, \ldots, \overline{y_n})$ which in T^{eq} is equivalent to $\varphi(\pi_{E_1}(\overline{y}_1), \ldots, \pi_{E_n}(\overline{y}_n))$.

Corollary 1.33. In M^{eq} there are no new definable relations on M. In particular, if M is stable / totally transcendental / NIP / ω-categorical then so is M^{eq}.

Example 1.34. If $H < G$ is 0-definable subgroup, then the cosets in G/H are the elements of the sort corresponding to $aE_Hb \iff ab^{-1} \in H$.

Furthermore if $H \trianglelefteq G$ is a normal subgroup then G/H is an interpretable group in G and is stable etc. if G is.

2 ω-categorical groups

Definition 2.1. A countable $\mathcal{L}$-structure M is called **ω-categorical** iff $\mathrm{Aut}(M)$ has only finitely many orbits on M^n for each n.

Example 2.2.

- $(\mathbb{Q}, <)$ is ω-categorical:

  Take $a_1 < \ldots < a_n$, and $b_1 < \ldots < b_n$, $a_i, b_i \in \mathbb{Q}$. Put $\varphi(a_i) := b_i$. Since $\mathbb{Q}$ is dense, φ can be extended to an automorphism of $\mathbb{Q}$.

- The random graph is ω-categorical.

- Vector spaces over a finite field K viewed as $(V, +, 0, \lambda_k : k \in K)$, where λ_k denotes scalar multiplication by k.

  Note that for an infinite field two elements can be linearly dependent in infinitely many ways. Hence vector spaces of an infinite field are not ω-categorical.

Remark 2.2.11.

(i) M is ω-categorical iff there is a unique countable structure elementarily equivalent[a] to M (up to isomorphism).

(ii) M is ω-categorical iff for any finite set $A \subseteq M$, $\mathrm{Aut}_A(M)$[b] has only finitely many orbits.

Exercise

(iii) If M is ω-categorical and $A \subseteq M^n$ is invariant under $\mathrm{Aut}_B(M)$ for some finite set $B \overset{\text{finite}}{\subseteq} M$, then A is B-definable.

Exercise

In particular if G is ω-categorical, then all characteristic subgroups are $\varnothing$-definable.

[a] $\mathcal{L}$-structures M, N are **elem** ... the same sentences, i.e. $\{\varphi | M \models$ ...

... enotes the **point** ...

Definition 2.3. A group G is called **locally finite** iff every finite subset generates a finite subgroup.

It is called **uniformly locally finite** iff for all $n \in \mathbb{N}$, there is a bound $k \in \mathbb{N}$, such that for all $a_1, \ldots, a_n \in G$, we have $|\langle a_1, \ldots, a_n \rangle| \leqslant k$.

In particular, a (uniformly) locally finite group is torsion (of bounded exponent).

Lemma 2.4. If G is an ω-categorical group, then G is uniformly locally finite.

Proof. Any automorphism of G fixing $a_1, \ldots, a_n$ fixes $\langle a_1, \ldots, a_n \rangle$ pointwise, hence $\langle a_1, \ldots, a_n \rangle$ is finite, as otherwise $\mathrm{Aut}_{a_1,\ldots,a_n}(G)$ has infinitely many orbits on M, one for each $x \in \langle a_1, \ldots, a_n \rangle$ (cf. Remark 2.2.11).

Since there are only finitely many orbits on n-tuples, and n-tuples in the same orbit generate isomorphic subgroups, the maximal bound works for all n-tuples. $\square$

So far we have not used stability; now we'll add this assumption.

Theorem 2.5. If G is ω-categorical and stable, then the **connected component**

$$G^0 := \bigcap \{H < G | H \text{ definable (with parameters) of finite index}\}$$

is $\varnothing$-definable and of finite index.

Proof. If $H < G$ is definable (with parameters) and of finite index, then $H^0 := \bigcap_{\varphi \in \mathrm{Aut}(G)} \varphi(H)$ is a finite intersection (by Baldwin-Saxl (1.18)) and hence of finite index in G.

Since H^0 is a characteristic subgroup, it is $\varnothing$-definable. There are only finitely many such subgroups (cf. Remark 2.5.12 (i)), hence G^0 is $\varnothing$-definable and of finite index. $\square$

Remark 2.5.12. (i) An ω-categorical group has only finitely many characteristic subgroups:

If $H \lhd_{\text{char}} G$, $\tilde{G} := \mathrm{Aut}(G)$, then $x^{\tilde{G}} \in H$ or $x^G \cap H = \varnothing$ for all $x \in G$. Since there are only finitely many 1-orbits, the claim follows.

(ii) An ω-categorical stable group G contains minimal normal subgroups and any normal subgroup contains a minimal one:

There are only finitely many $\mathrm{Aut}(G)$-orbits on $G \times G$. Hence there is some $k \in \mathbb{N}$ such that for $x \in \langle y \rangle^G$ we have $x = y^{g_1} \cdot y^{g_i}$ for some $i \leqslant k$. Hence all normal subgroups of the form $\langle a^G \rangle$ are uniformly definable,

$$\langle a^G \rangle = \{a^{g_1} \cdot \ldots \cdot a^{g_i} | g_i \in G, i \leqslant k\}.$$

By the Trivial Chain Condition (1.13.4), there is a minimal one.

(iii) A stable group does not contain subgroups which are unbounded direct products of non-abelian groups.

If $H_1 \times \ldots \times H_k \leqslant G$, $h_i \in H_i \backslash Z(H_i)$, then $\bigcap_{j \neq i} \mathrm{Cen}(h_j) \geqslant H_i$ and $H_i \nleqslant \bigcap_{j \leqslant k} \mathrm{Cen}(h_j)$.

A Tutorial

[Tutorial 01, 2024-04-16]

A.1 Recap

A.1.1 Group actions

Definition A.1. Let G be a group and X a set. A **group action** $G \curvearrowright X$ is a group homomorphism $\pi \colon G \to \mathrm{Sym}(X)$. For $g \in G, x \in X$ we will write $g.x := \pi(g)(x)$.

- The action is **transitive** iff $\forall x, x' \in X.\ \exists g \in G.\ gx = x'$.
- The action is **faithful** iff π is surjective.
- The action is **free** iff no non-trivial element has a **fixpoint**, i.e. $\forall g \in G \backslash \{1\}, x \in X.\ g.x \neq x$.
- The action is **regular** iff it is transitive and free.
- The **stabilizer** of $x \in X$ is the subgroup $G_x := \{g \in G : g.x = x\}$.
- The **orbit** of $x \in X$ is $G.x := \{g.x | g \in G\}$.

Definition A.2. For a subgroup $H \leqslant G$ the **index** of H in G, $|G : H|$ is defined as $|\{gH / g \in G\}|$.

Theorem A.3 (Orbit stabilizer theorem)**.** Let $G \curvearrowright X$, $x \in X$. Then $|G.x| = |G : G_x|$.

Proof. The map

$$\begin{aligned} \varphi \colon G.x &\longrightarrow G/G_x \\ g.x &\longmapsto gG_x \end{aligned}$$

is bijective. □

A.1.2 Nilpotent and solvable groups

Definition A.4. Let G be a group. The **commutator** or **derived subgroup** of G, denoted $[G, G]$ or G', is defined as $\{[g, g'] : g, g' \in G\}$. $[G, G]$ is the smallest subgroup of G such that G/G' is abelian.

We recursively define $G^{(0)} := G$ and $G^{(n+1)} := [G^{(i)}, G^{(i)}]$.

We say that G is **solvable** if $G^{(n)} = 1$ for some $n \in \mathbb{N}$.

Proposition A.5. G is solvable iff we have a sequence $1 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_m = G$ such that H_{i+1}/H_i is abelian.

Example A.6. Let K be a field, $\mathrm{AGL}_1(K) := \{x \mapsto ax+b | a \neq 0, b \in K\} \cong K \rtimes K$ the group of affine transformations.

We have $\mathrm{AGL}_1(K)/K^+ \cong K^+$, so $1 \trianglelefteq K^+ \trianglelefteq \mathrm{AGL}_1(K)$.

Definition A.7. A **linear algebraic group** of a field k is a subgroup of $\mathrm{GL}_1(K)$, e.g. $\mathrm{SL}_1(K)$. For such a group, a Borel subgroup is a maximal closed connected solvable group.

Example A.8.
- A Borel subgroup for $\mathrm{GL}_n(K)$ is the group of upper triangular matrices.
- A Borel subgroup for $\mathrm{SL}_n(K)$ is the group of upper **unitriangular** matrices (i.e. upper triangular matrices with only 1 on the diagonal).

Definition A.9. Let G be a group. We define $G^{[n]}$ inductively be $G^{[0]} := G$, $G^{[n]} := [G, G^{[n-1]}]$.

We say that G is (n-step) **nilpotent** iff there exists $n \in \mathbb{N}$ (minimal) such that $G^{[n]} = 1$.

Nilpotent groups are solvable.

Proposition A.10. The following are equivalent
- G is nilpotent.
- There exists a finite ascending central series, $\zeta_0(G) = 1 \trianglelefteq \zeta_1(G) = Z(G) \trianglelefteq \ldots \trianglelefteq \zeta_n(G) = G$ with $\zeta_{i+1}(G) = \{g \in G : g\zeta_i(G) \in Z(G/\zeta_i(G))\}$, that is $\zeta_{i+1}(G)$ is the subgroup such that $\zeta_{i+1}(G)/\zeta_i(G) = Z(G/\zeta_i(G))$.

Example A.11.
- For a field K, the group of upper unitriangular matrices is nilpotent.
- The **quaternion group** $Q_8 = \langle a, b | a^4 = e, a^2 = b^2, ba = a^{-1}b\rangle$ is nilpotent.

If a group is nilpotent / solvable, then its subgroups have the same property.

A.1.3 The Compactness Theorem

Definition A.12. Let $\mathcal{L}$ be a first order languages and Σ a set of sentences in the language $\mathcal{L}$. We say that Σ is **satisfiable** iff there exists an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \sigma$ for all $\sigma \in \Sigma$ (or short $\mathcal{M} \models \Sigma$).

We say that Σ is **finitely satisfiable** iff every finite subset $\Sigma' \subseteq \Sigma$ is satisfiable.

Theorem A.13 (Compactness Theorem)**.** Σ is satisfiable iff it is finitely satisfiable.

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