Stable Groups

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Contents

These are my notes on the lecture "Stable Groups", taught by Prof. Dr. Dr. Katrin Tent in the summer term 2024 at the University Münster.

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[Lecture 01, 2024-04-15]

A background in model theory is helpful but not necessary. Some group theory is required (usually covered in linear algebra and a first algebra course).

The lecture starts at 08:25.

The book by Prof. Tent is available on learnweb.

There will be an oral exam. For a type II course, one needs to do nothing.

The main point is to see, how model theoretic properties influence algebraic properties.

1 Introduction

Definition 1.1. An infinite $\mathcal{L}$-structure M is **minimal** iff for every formula $\varphi(x) \in \mathcal{L}(M)$, the set defined by φ, $\varphi(M) := \{a \in M | M \models \varphi(a)\}$ is finite or cofinite.

Example 1.2.
- $(\mathbb{Q}, +, \cdot)$ is not minimal, consider for example the formula $\varphi(x) := \exists y.\ x = y^2$
- $(\mathbb{C}, +, \cdot)$ is minimal.

Recall the **orbit equation**: If $G \curvearrowright X$ is transitive, then there is a natural bijection

$$\begin{aligned} {}^{G}\!/\!_{G_x} &\longrightarrow X \\ hG_x &\longmapsto h \cdot x \end{aligned}$$

where for $x \in X$, $G_x := \{g \in G : gx = x\} \leqslant G$ is the **stabilizer** of x in G and $G \cdot x := \{gx : g \in G\} \subseteq X$ is the **orbit** of x under G.

Theorem 1.3 (Reineke)**.** Minimal groups are abelian.

Proof. Let G be a minimal group.

Since G is minimal, all proper definable[1] subgroups are finite by minimality: If $H \lneq G$ is a proper definable subgroup, then for $a \notin H$, the coset $a \cdot H$ is also definable and disjoint from H.

Suppose that G is not abelian. Then the center[2] $Z(G)$ is finite. Furthermore, every element of the group must have finite order, since $\langle a \rangle \leqslant Z(\mathrm{Cen}(a))$.[3] (Note that $\langle a \rangle$ is not definable in general).

[1] A **definable** subgroup is a subgroup, that can be defined by a formula.
[2] The **center** is defined as $Z(G) := \{x \in G : \forall y.\ xy = yx\}$.
[3] The **centralizer** of a is the set of all elements commuting with a.

Consider the conjugacy class $a^G := \{a^g : g \in G\}$, where $a^g := g^{-1}ag$. Then for $a \in Z(G)$,[4] we have $|a^G| = |{}^G\!/\!_{G_a}|$, where $G_a = \mathrm{Cen}(a)$. In particular, for $a \notin Z(G)$, the conjugacy class a^G is infinite. Since by minimality there can not be two disjoint infinite conjugacy classes, we get $G = a^G \cup Z(G)$ for all $a \notin Z(G)$. Thus any $a, b \in G \backslash Z(G)$ are conjugate, so a, b have the same finite order and $|\mathrm{Cen}(a)| = |\mathrm{Cen}(b)|$.

If all elements have order 2, the group is abelian, since $a^{-1}b^{-1}ab = abab = 1$ in this case.

If all $a \in G\backslash Z(G)$ have order 2, then again G is abelian: Let $c \in Z(G)$, then $ca \notin Z(G)$, so $1 = (ac)^2 = acac = a^2c^2 = c^2$, i.e. the elements in $Z(G)$ also have order 2.

Now let $a \in G\backslash Z(G)$ be not of order two. Then $a^2 \neq 1$ and $a, a^{-1} \notin Z(G)$ are conjugate under some $g \in G$, i.e. $b^{-1}ab = a^{-1}$, hence $b^{-2}ab^2 = a$, $b^2 \in \mathrm{Cen}(a)$. So $a \in \mathrm{Cen}(b^2)\backslash \mathrm{Cen}(b)$. Clearly $\mathrm{Cen}(b) \leqslant \mathrm{Cen}(b^2)$ and a witnesses that this is a proper subgroup. So $|\mathrm{Cen}(b)| \neq |\mathrm{Cen}(b^2)|$, hence $b^2 \in Z(G)$. It follows that $H = {}^G\!/\!_{Z(G)}$ is an elementary abelian 2-group in which all non-trivial elements are conjugate, i.e. $|H| = 2$[5] and so G is finite. □

We want to generalize this.

Definition 1.4. An $\mathcal{L}$-structure M is **stable** iff there are no $M \preceq \tilde{M}$,[a] $\mathcal{L}(\tilde{M})$-formula $\varphi(\overline{x}, \overline{y})$ and tuples $\overline{a_i}, \overline{b_j} \in \tilde{M}$ such that $\tilde{M} \models \varphi(\overline{a_i}, \overline{b_j})$ iff $i < j$.

[a] elementary extension

Example 1.5. Let $M = (\mathbb{Z}, +, \cdot, 0, 1)$, $a_i = i = b_i$ and

$$\varphi(x, y) \longleftarrow \ulcorner \exists z_1, \ldots, z_4.\ x + z_1^2 + \ldots + z_4^2 = y \urcorner.$$

Then $M \models \varphi(a, b_j)$ iff $i \leqslant j$. So M is not stable.

Algebraically closed fields are stable.

Lemma 1.6. If M is a stable and non-empty **semigroup**[a] with right- and **left-cancellation**[b] (alternatively: left-cancellation and a **right neutral element**[c]), then M is a group.

[a] associative operation
[b] $ax = ay \implies x = y$
[c] $\forall a.\ ae = a.$

[4] Note that for $a \in Z(G)$, we have $a^G = \{a\}$.
[5] Conjugation is not too interesting in abelian groups.

Proof. The formula $\varphi(x,y) \longleftrightarrow \ulcorner \exists z.\ x \cdot z = y \urcorner$, is satisfied by (a^n, a^m) if $n < m$. By stability, this can not be an if and only if. So there must be some $m > n$, such that $M \models \varphi(a^n, a^m)$. I.e. there is some $b \in M$ such that $a^n = a^{n+p}b$, where $m = n + p$. Put $e = a^p b$. This is a left-neutral element: For $c \in M$ we have $a^n c = a^n ec$, hence $c = ec$ by left-cancellation.

By symmetry (or assumption), there exists a right-neutral element f, and since $e = ef = f$, e is neutral.

Furthermore
$$e = a^p b \overset{p \geq 0}{=} a(a^{p-1}b),$$
so a has an inverse. □

Remark 1.6.1. The assumptions are necessary since a semigroup with $xy = y$ is not a group.

Corollary 1.7. If G is stable, then every non-empty definable subset closed under multiplication is a subgroup.

Similarly, every definable non-empty subring of a stable field is a subfield.

Remark 1.7.2. A stable group is a group whose theory is stable (not necessarily in the language of groups). The group may be a definable (or interpretable) structure inside another structure, e.g. $(K, +, \cdot, 0, 1)$ field, $G = \mathrm{GL}_n(K)$ or any other Chevalley group.

Definition 1.8. A **definable group action** (in some L-structure M) is given by a definable group G, a definable set X and a definable action $G \times X \to X$ (i.e. the graph of the action is a definable subset of $(G \times X) \times X$.

Example 1.9. Let $(K, +, \cdot, 0, 1)$ be a field. Then $\mathrm{GL}_n(K)$, K^n and the action $\mathrm{GL}_n(K) \curvearrowright K^n$ are definable.

Example 1.10. Consider $(\mathbb{Q}, +, \cdot, 0, 1)$. Then $A := [0, 1]$ is definable[a] and $\frac{1}{n}A \subsetneq A$. Hence it is not stable by the following lemma.

[a]this is non-trivial

Lemma 1.11. Let G be a stable group acting definably on a set X. If $A \subseteq X$ is definable and $g \in G$, then $g(A) \subseteq A$ iff $g(A) = A$.

Proof. If $g(A) \subsetneq A$, we get a proper descending sequence $A \supsetneq g(A) \supsetneq g^2(A) \supsetneq g^3(A) \supsetneq \ldots$ and the sequence g^i, $i < \omega$ is ordered by $\ulcorner xA \subsetneq yA \urcorner$. □

Recall:

Corollary 1.12 (of Lemma 1.11)**.** If G is stable, $A \subseteq G$ is definable and $g \in G$, then $A^g \leqslant A \iff A^g = A$.

Remark 1.12.3. This does not hold in general. Consider

$$H = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z} \right\} < \mathrm{GL}_2(\mathbb{Q})$$

and

$$g := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, h_m := \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}.$$

Then $gh_m g^{-1} = h_{2m}$, so $gHg^{-1} \lneq H$.

Definition 1.13. For a group G, a family of subsets $\{H_i\}_{i \in I}$ of M^k is called **uniformly definable** if there is a formula $\varphi(\overline{x}, \overline{y})$ and $\overline{a_i} \in M_i, i \in I$ such that $\varphi(M^k, \overline{a_i}) = H_i$.

For example, the centralizers of elements are uniformly definable.

Remark 1.13.4. If G is stable, the **Trivial Chain Condition** holds for uniformly definable subsets and subgroups, i.e. descending chains are finite:

For every uniformly definable family $H_i, i \in I$, there is some $n < \omega$ such that every properly descending (resp. ascending) chain $H_{i_1} \lneq H_{i_2} \lneq H_{i_3} \lneq \ldots$ has length at most n. This n depends only on the formula, not on the parameters of the form of the definable family.

Definition 1.14. A formula $\varphi(\overline{x}, \overline{y})$ has the **independence property** (**IP**) iff there are $\overline{a_i}, i < \omega$, such that for all $A \subseteq \omega$, the set $\{\varphi(\overline{x}, \overline{a_i}) | i \in A\} \cup \{\neg\varphi(\overline{x}, \overline{a_i}) | i \notin A\}$ is consistent.

relate to wikipedia definition (use compactness), see proof of Lemma 1.17

A theory is called **NIP** iff no consistent formula has IP.

Example 1.15.

- The random graph (Radograph) has IP.
- $(\mathbb{C}, +, \cdot, 0, 1)$ is NIP.

Lemma 1.16. If T is stable, then T is NIP.

Proof. If $\varphi(\overline{x}, \overline{y})$ has IP, $\overline{a_i} \in M, i < \omega$, $M \models T$, then there are $\tilde{M} \succeq M$, and $b_i \in \tilde{M}$ such that $\tilde{M} \models \varphi(\overline{b_i}, \overline{a_j})$ iff $i < j$, which is a contradiction to stability. □

The reverse direction does not hold, since for example the real numbers have NIP, but are not stable.

Lemma 1.17. Let G be a NIP group. Then finite intersections of uniformly definable subgroups are uniformly bounded, i.e. for every formula $\varphi(x, \overline{y})$ there is $n < \omega$ such that if $H_i = \varphi(G, \overline{a_i})$, $i = 1, \ldots, m$ are subgroups, then

$$\bigcap_{i \leqslant m} H_i = \bigcap_{j=1}^{n} H_{i_j}.$$

Proof. Suppose not. Then for all $n < \omega$ there is a uniformly definable family of subgroups $H_1, \ldots, H_n$ such that $\bigcap H_i \lneq \bigcap_{\substack{i=1 \\ i \neq j}}^{n} H_i$ for any $1 \leqslant j \leqslant n$.

So there is some $b_j \in \left(\bigcap_{i \neq j} H_i\right) \backslash H_j$, $j \leqslant n$.

For $I \subseteq \{1, \ldots, n\}$ put $b_I := \prod_{i \in I} b_i$. Then $G \models \varphi(b_j, \overline{a_i})$ iff $i \notin J$. Since n was arbitrary, this shows that $\varphi(x, y)$ has IP: Let $A \subseteq \omega$ be any subset. By the Compactness Theorem (A.14) it suffices to show that every finite subset of $\{\varphi(x, \overline{a_i}) | i \in A\} \cup \{\neg\varphi(x, \overline{a_i}) | i \notin A\}$ is consistent. This holds, since for every $I \overset{\text{finite}}{\subseteq} \omega$,

$$G \models \{\varphi(b_{I \backslash A}, \overline{a_i}) | i \in A \cap I\} \cup \{\neg\varphi(b_{I \backslash A}, \overline{a_i}) | i \in I \backslash A\}.$$

□

Proposition 1.18 (Baldwin-Saxl)**.** If G is stable, then for every formula $\varphi(\overline{x}, \overline{y})$, there is $n < \omega$ (depending only on φ) such that for subgroups $H_i = \varphi(G, \overline{a_i})_{i \in I}$, we have

$$\bigcap_{i \in I} H_i = \bigcap_{j=1}^{n} H_{i_j}$$

for some $i_j \in I$, i.e. arbitrary intersections of uniformly definable subgroups are definable.

Proof. By Lemma 1.17 intersections of finitely many H_i are uniformly definable. By the Trivial Chain Condition (1.13.4) applied to these uniformly definable intersections, there is a minimal group H is this family, i.e. $H = \bigcap H_i = \bigcap_{j=1}^{n} H_{i_j}$ with $n < \omega$ from Lemma 1.17. □

Corollary 1.19. If G is stable and $A \leqslant G$ arbitrary, then $\mathrm{Cen}(A) = \bigcap_{a \in A} \mathrm{Cen}(a) = \{g \in G | \forall a \in A.\ [g, a] = 1\}$ is definable.

Example 1.20. By Sela's Theorem the free groups F_n are stable. For $w \in F_k$, $\mathrm{Cen}(w)$ is cyclic, so $n = 2$.

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Remark 1.20.5. Since the formula $\ulcorner xa = ax \urcorner$ is quantifier-free, Corollary of Baldwin-Saxl (1.19) holds in all subgroups of stable groups.

For example $\mathrm{Sym}_{\mathrm{fin}}(\omega)$, the group of permutations of ω with finite support (i.e. moving only finitely many elements) can never be a subgroup of a stable group, since centralizers can become arbitrarily small.

Definition 1.21. Let T be arbitrary and $\varphi_s(\overline{x}), s \in 2^{<\omega}$ consistent formulae.[a]

Then

(i) the $\varphi_s(\overline{x})$ form a **binary tree of consistent formulae** iff

$$T \vdash \forall \overline{x}(\varphi_{s^\frown 0}(\overline{x}) \vee \varphi_{s^\frown 1}(\overline{x}) \to \varphi_s(\overline{x}))$$

and

$$T \vdash \forall \overline{x} \neg(\varphi_{s^\frown 0}(\overline{x}) \wedge \varphi_{s^\frown 1}(\overline{x})).$$

(ii) T is called **totally transcendental** (or **ω-stable** iff $\mathcal{L}$ is countable) iff there is no binary tree of consistent formulae.

[a] Here "consistent" means that the family is consistent along every path, i.e. for every $s \in 2^\omega$, $\{\varphi_{s|_n} : n \in \omega\}$ is consistent. The entire family may be inconsistent.

Example 1.22. Let G be a group, and $H_i, i < \omega$ an infinite descending chain of subgroups $H_i \gneqq H_{i+1}$, then we get a binary tree (subset vs. coset). So totally transcendental is much stronger than stable.

Proposition 1.23. If G is totally transcendental, there is no infinite properly descending chain of definable subgroups.

Proof. Otherwise we get a binary tree. $\square$

[Lecture 03, 2024-04-22]

Corollary 1.24.

(i) In a totally transcendental group G every intersection of definable subgroups is definable. In particular, there is a minimal definable subgroup G^0 of finite index in G, the **connected component** of G.

(ii) If G is totally transcendental, every injective definable endomorphism of G is surjective, i.e. an automorphism of G.

(iii) If G is ω-stable, abelian and torsion free, then G is divisible[a].

[a]An abelian group A is **divisible** iff $\forall a \in A.\ \forall n \in \mathbb{N}.\ \exists b \in A.\ n \cdot b = a$, i.e. iff $G \cong \bigotimes_{i \in I} \mathbb{Q}$.

Proof. (i) Clear.

(ii) Suppose that $s\colon G \hookrightarrow G$ is definable but not surjective. Then $s^i(G)$ is a proper descending sequence of definable subgroups $\lightning$.

(iii) Note that the map $g \mapsto n \cdot g$ is definable and injective.[6]

□

Remark 1.24.6. If G is stable, then for any formula $\varphi(x, \overline{y})$ the group

$$G^0(\varphi) = \bigcap \{\varphi(G, \overline{a}) | \varphi(G, \overline{a}) \leqslant G, [G : \varphi(G, \overline{a})] < \infty\}$$

is a definable subgroup of finite index by Baldwin-Saxl (1.18), the **φ-connected component of G**.

In particular, we'll be interested in the case

$$\varphi(x, y) \longleftrightarrow \ulcorner xy = yx \urcorner.$$

Definition 1.25. A group G is called **centralizer connected** iff $G = G^0(xy = yx)$, i.e. iff for all $a \in G \backslash Z(G)$ the index $[G : \mathrm{Cen}(a)]$ is infinite.

Lemma 1.26. If G is centralizer connected, $A \subseteq G$ finite and A normalized by[a] G, then $A \subseteq Z(G)$.

[a]For $A, B \leqslant G$ we say that A is **normalized** by B iff $\forall b \in B.\ A^b = A$, i.e. $B \leqslant N_G(A)$.

Proof. If a^G is finite, then $a \in Z(G)$, since $|G : \mathrm{Cen}(a)| = |a^G|$. □

Remark 1.26.7. This does not depend on stability.

[6]Warning: $g \mapsto n \cdot g$ is not uniformly definable.

Proposition 1.27. If G is stable[a] and $\{[g,h] | g,h \in G\}$ finite, then G is virtually abelian.[b]

[a] The assumption of G being stable can be removed.

[b] A group is called **virtually abelian** or **abelian-by-finite** iff $Z(G)$ has finite index in G.

Proof. For every $g \in G$, the set $\{[g,h] : h \in G\}$ is finite. Hence g^G is finite, so $|G : \mathrm{Cen}(g)|$ is finite. By the Corollary of Baldwin-Saxl (1.19), we have $Z(G) = \bigcap_{i \leqslant n} \mathrm{Cen}(g_i)$ for some $n \in \mathbb{N}$, and this has finite index. □

Proposition 1.28. If G is centralizer connected with finite center, then $Z(G) = \zeta_2(G)$, i.e. $Z(G/Z(G)) = \{1\}$.

Corollary 1.29. If G is centralizer connected, infinite and nilpotent, then $Z(G)$ is infinite.

Proof of Proposition 1.28. Recall that $\zeta_2(G) = \{g \in G | gZ(G) \in Z(G/Z(G))\}$. So for all $g \in \zeta_2(G)$, $h \in G$ we have $[g,h] \in Z(G)$.

Since $Z(G)$ is finite, we get for $g \in \zeta_2(G)$ that the orbit g^G is finite, so $[G : \mathrm{Cen}(g)]$ is finite. Hence $g \in Z(G)$, since G is centralizer connected. □

Remark 1.29.8. If G is nilpotent, $1 \neq N \trianglelefteq G$, then $N \cap Z(G) \neq \{1\}$:

Suppose $n \in (N \cap \zeta_i(G)) \backslash \{1\}$ with i minimal. If $i > 1$, then there exists $g \in G$ such that $1 \neq [g,n] \in \zeta_{i-1}(G) \cap N$.

Lemma 1.30. If G is nilpotent, centralizer connected and $N \trianglelefteq G$ infinite[a], then $N \cap Z(G)$ is infinite.

[a] not necessarily definable

Proof. If $N \leqslant Z(G)$ this is trivial. Otherwise $N \cap Z(G) \neq \{1\}$. If $1 \neq n \in N \cap \zeta_2(G) \backslash Z(G)$, then n^G is infinite and $n^{-1} \cdot n^G = [n,G] \subseteq Z(G) \cap N$ is infinite. □

Remark 1.30.9. If G is nilpotent, then for any subgroup $H \lneq G$ we have $H \lneq N_G(H)$ (cf. Sheet 1, Exercise 1 (B.1.1)).

Theorem 1.31. If G is stable, nilpotent, and $H < G$ definable of infinite index, then H has infinite index in $N_G(H)$.

Proof. Let $Z := Z(G)$. If $[ZH : H]$ is infinite, the claim is clear.

Now we use induction on the length of the central series: If $[ZH : H]$ is finite, then $[G : ZH] = [G/Z : ZH/Z]$ is infinite. By the inductive assumption ${}^{ZH}\!/_{Z}$ has infinite index in $N_{G/Z}(ZH/Z)$, hence ZH has infinite index in $N_G(ZH)$. We have

$$H \leqslant ZH \leqslant N_G(H) \leqslant N_G(ZH) =: N.$$

By Baldwin-Saxl (1.18) it is

$$\bigcap_{n \in \mathbb{N}} H^n = H^{n_1} \cap \ldots \cap H^{n_2} =: H^0$$

for some $l \in \mathbb{N}$.

Since $[ZH : H]$ is finite, H^0 has finite index in H and $N_G(ZH) \leqslant N_G(H^0)$. We obtain

$$H^0 \leqslant H \leqslant ZH,$$

where each step is of finite index. Hence

$$\left({}^{H}\!/_{H^0}\right)^N \subseteq {}^{ZH}\!/_{H^0}$$

is finite. Therefore $N_N(H)$ has finite index in N. Since $N_N(H) \leqslant N_G(H) \leqslant N$, the claim follows. □

[Lecture 04, 2024-04-25]

Remark 1.31.10. Note that when taking a quotient by a $\varnothing$-definable subgroup, e.g. $G/Z(G)$ in the proof of Theorem 1.31, the elements of the quotient are not elements of our structure. However the quotient is **interpretable** in G, i.e. equality up $Z(G)$ can be written as a formula in our language. We call elements of such an interpretable structure **virtual elements**.

More generally if E is a $\varnothing$-definable equivalence relation on M^n for some $\mathcal{L}$-structure M, $n \in \mathbb{N}$, we can extend the structure by a new **sort** of elements, whose elements are the equivalence classes modulo E. We extend the language $\mathcal{L}$ to a language $\mathcal{L}^{\text{eq}}$ by adding for each such equivalence relation E a new sort and a new n-ary function symbol $\pi_E \colon M^n \to {}^{M^n}\!/_{E}$.

Lemma 1.32. For every $\mathcal{L}^{\text{eq}}$-formula $\varphi(x_1, \ldots, x_n)$, where x_1 is of the sort N^{n_i}/E_i, there is an $\mathcal{L}$-formula $\psi(\overline{y_1}, \ldots, \overline{y_n})$ which in T^{eq} is equivalent to $\varphi(\pi_{E_1}(\overline{y}_1), \ldots, \pi_{E_n}(\overline{y}_n))$.

Corollary 1.33. In M^{eq} there are no new definable relations on M. In particular, if M is stable / totally transcendental / NIP / ω-categorical then so is M^{eq}.

Example 1.34. If $H < G$ is 0-definable subgroup, then the cosets in G/H are the elements of the sort corresponding to $aE_H b \iff ab^{-1} \in H$.

Furthermore if $H \trianglelefteq G$ is a normal subgroup then G/H is an interpretable group in G and is stable etc. if G is.

2 ω-categorical groups

Definition 2.1. A countable $\mathcal{L}$-structure M is called **ω-categorical** iff $\mathrm{Aut}(M)$ has only finitely many orbits on M^n for each n.

Example 2.2.

- $(\mathbb{Q}, <)$ is ω-categorical:

  Take $a_1 < \ldots < a_n$, and $b_1 < \ldots < b_n$, $a_i, b_i \in \mathbb{Q}$. Put $\varphi(a_i) := b_i$. Since $\mathbb{Q}$ is dense, φ can be extended to an automorphism of $\mathbb{Q}$.
- The random graph is ω-categorical.
- Vector spaces over a finite field K viewed as $(V, +, 0, \lambda_k : k \in K)$, where λ_k denotes scalar multiplication by k.

  Note that for an infinite field two elements can be linearly dependent in infinitely many ways. Hence vector spaces of an infinite field are not ω-categorical.

Remark 2.2.11.

(i) M is ω-categorical iff there is a unique countable structure elementarily equivalent[a] to M (up to isomorphism).

(ii) M is ω-categorical iff for any finite set $A \subseteq M$, $\mathrm{Aut}_A(M)$[b] has only finitely many orbits.

Exercise

(iii) If M is ω-categorical and $A \subseteq M^n$ is invariant under $\mathrm{Aut}_B(M)$ for some finite set $B \overset{\text{finite}}{\subseteq} M$, then A is B-definable.

Exercise

In particular if G is ω-categorical, then all characteristic subgroups are $\varnothing$-definable.

[a] $\mathcal{L}$-structures M, N are **elem** … the same sentences, i.e. $\{\varphi | M \models$ …

[b] … enotes the **point** …

Definition 2.3. A group G is called **locally finite** iff every finite subset generates a finite subgroup.

It is called **uniformly locally finite** iff for all $n \in \mathbb{N}$, there is a bound $k \in \mathbb{N}$, such that for all $a_1, \ldots, a_n \in G$, we have $|\langle a_1, \ldots, a_n \rangle| \leqslant k$.

In particular, a (uniformly) locally finite group is torsion (of bounded exponent).

Lemma 2.4. If G is an ω-categorical group, then G is uniformly locally finite.

Proof. Any automorphism of G fixing $a_1, \dots, a_n$ fixes $\langle a_1, \dots, a_n \rangle$ pointwise, hence $\langle a_1, \dots, a_n \rangle$ is finite, as otherwise $\mathrm{Aut}_{a_1,\dots,a_n}(G)$ has infinitely many orbits on M, one for each $x \in \langle a_1, \dots, a_n \rangle$ (cf. Remark 2.2.11).

Since there are only finitely many orbits on n-tuples, and n-tuples in the same orbit generate isomorphic subgroups, the maximal bound works for all n-tuples. □

So far we have not used stability; now we'll add this assumption.

Theorem 2.5. If G is ω-categorical and stable, then the **connected component**

$$G^0 := \bigcap \{H < G | H \text{ definable (with parameters) of finite index}\}$$

is $\varnothing$-definable and of finite index.

Proof. If $H < G$ is definable (with parameters) and of finite index, then $H^0 := \bigcap_{\varphi \in \mathrm{Aut}(G)} \varphi(H)$ is a finite intersection (by Baldwin-Saxl (1.18)) and hence of finite index in G.

Since H^0 is a characteristic subgroup, it is $\varnothing$-definable. There are only finitely many such subgroups (cf. Remark 2.5.12 (i)), hence G^0 is $\varnothing$-definable and of finite index. □

Remark 2.5.12.

(i) An ω-categorical group has only finitely many characteristic subgroups:

If $H \lhd_{\text{char}} G$, $\tilde{G} := \mathrm{Aut}(G)$, then $x^{\tilde{G}} \subseteq H$ or $x^G \cap H = \varnothing$ for all $x \in G$. Since there are only finitely many 1-orbits, the claim follows.

(ii) An ω-categorical stable group G contains minimal normal subgroups and any normal subgroup contains a minimal one:

There are only finitely many $\mathrm{Aut}(G)$-orbits on $G \times G$. Hence there is some $k \in \mathbb{N}$ such that for $x \in \langle y \rangle^G$ we have $x = y^{g_1} \cdot y^{g_i}$ for some $i \leqslant k$. Hence all normal subgroups of the form $\langle a^G \rangle$ are uniformly definable,

$$\langle a^G \rangle = \{a^{g_1} \cdot \dots \cdot a^{g_i} | g_i \in G, i \leqslant k\}.$$

By the Trivial Chain Condition (1.13.4), there is a minimal one.

(iii) A stable group does not contain subgroups which are unbounded direct products of non-abelian groups.

If $H_1 \times \ldots \times H_k \leqslant G$, $h_i \in H_i \backslash Z(H_i)$, then $\bigcap_{j \neq i} \mathrm{Cen}(h_j) \geqslant H_i$ and $H_i \nleqslant \bigcap_{j \leqslant k} \mathrm{Cen}(h_j)$.

By the Corollary of Baldwin-Saxl (1.19), there is a bound on k depending only on $\mathrm{Th}(G)$.

(iv) Every finite simple group is 2-generated.

[Lecture 05, 2024-04-29]

Theorem 2.6 (Baw-Cherlin-Macintyre, Felgner)**.** An ω-categorical stable group G is virtually nilpotent.

TODO: Proof

[Lecture 06, 2024-05-02]

TODO

and 6)

3 Morley Rank

The **Morley rank** is a notion of dimension on definable sets, similarly to the algebraic dimension of an algebraic variety (and agrees with it in this context).

In this section let T always denote a complete theory with infinite models.

Definition 3.1. Let $\varphi(\overline{x})$ be an $\mathcal{L}(\mathcal{M})$-formula, $\mathcal{M} \models T$ very saturated.

(i) $\mathrm{MR}(\varphi) \geqslant 0$ if φ is consistent (i.e. $\varphi(M) \neq \varnothing$).

(ii) $\mathrm{MR}(\varphi) \geqslant \beta + 1$ if there is an infinite family of formulae $\varphi_i, i < \omega$ such that $\varphi_i \to \varphi$, $\varphi_i(M) \cap \varphi_j(M) = \varnothing$ for $i \neq j$ and $\mathrm{MR}(\varphi_i) \geqslant \beta$ for all $i < \omega$.

(iii) $\mathrm{MR}(\varphi) \geqslant \lambda$ for limit ordinals λ if $\mathrm{MR}(\varphi) \geqslant \alpha$ for all $\alpha < \lambda$.

If φ is inconsistent, put $\mathrm{MR}(\varphi) = -\infty$. If $\mathrm{MR}(\varphi) \geqslant \alpha$ for all $\alpha \in \mathrm{Ord}$, put $\mathrm{MR}(\varphi) := \infty$. If $\mathrm{MR}(\varphi) \geqslant \alpha$, $\mathrm{MR}(\varphi) \ngeqslant \alpha + 1$ put $\mathrm{MR}(\varphi) = \alpha$.

[Lecture 07, 2024-05-06]

Remark 3.1.13. (i) It is $\mathrm{MR}(\varphi) = 0$ iff $\varphi(\mathcal{M})$ is finite (in any model of T).

(ii) If for all $\mathcal{M} \models T$ we have $\varphi(\mathcal{M}) \subseteq \psi(\mathcal{M})$ (equivalently $T \vdash \varphi \to \psi$), then $\mathrm{MR}(\varphi) \leqslant \mathrm{MR}(\psi)$.

(iii) If $\mathrm{MR}(\varphi) = \alpha$ and $\beta \leqslant \alpha$ then there exists some ψ with $T \vdash \psi \to \varphi$ and $\mathrm{MR}(\psi) = \beta$.

Lemma 3.2. $\mathrm{MR}(\varphi \wedge \psi) = \max\{\mathrm{MR}(\varphi), \mathrm{MR}(\psi)\}$.

Proof. By Remark 3.1.13 (ii). $\mathrm{MR}(\varphi \wedge \psi) \geqslant \max\{\mathrm{MR}(\varphi), \mathrm{MR}(\psi)\}$. We show by induction on α, that if $\mathrm{MR}(\varphi \wedge \psi) \geqslant \alpha$, then $\max\{\mathrm{MR}(\varphi), \mathrm{MR}(\psi)\} \geqslant \alpha + 1$. If $\mathrm{MR}(\varphi \wedge \psi) \geqslant \alpha + 1$, then there exists $(\varphi_i)_{i<\omega}$, such that $T \vdash \varphi_i \to (\varphi \wedge \psi)$ and $\mathrm{MR}(\varphi_i) \geqslant \alpha$ for all $i < \omega$. By inductive assumption, $\mathrm{MR}(\varphi_i \wedge \varphi) \geqslant \alpha$ or $\mathrm{MR}(\varphi_i \wedge \psi) \geqslant \alpha$ for each $i < \omega$. Hence for φ or ψ there exists infinitely many i, such that $\mathrm{MR}(\varphi_i \wedge \varphi) \geqslant \alpha$ or $\mathrm{MR}(\varphi_i \wedge \psi) \geqslant \alpha$, so $\max\{\mathrm{MR}(\varphi), \mathrm{MR}(\psi)\} \geqslant \alpha + 1$. $\square$

Remark 3.2.14. .

- φ, ψ are called **disjoint** (over all models) if $T \cup \{\varphi, \psi\}$ is inconsistent.
- If $\mathrm{MR}(\varphi) = \alpha$, then there exist only finitely many disjoint formulae $\varphi_1, \dots, \varphi_d$ with $T \vdash \varphi_i \to \varphi$ and $\mathrm{MR}(\varphi_i) = \alpha$.

  The **Morley degree**, $\mathrm{Mdeg}(\varphi)$, is defined to be the maximum of all such d.

Theorem 3.3. A theory T is totally transcendental iff every formula has a Morley rank.[a]

[a] i.e. $\mathrm{MR}(\varphi) \neq \infty$

Proof. "$\Longrightarrow$" Any formula without a Morley rank can be decomposed into an infinite binary tree.

"$\Longleftarrow$" If $(\varphi_s)_{s \in {}^{<\omega}2}$ is a binary tree of consistent formulae, such that φ_s is of minimal Morley rank and Morley degree, then $\varphi_{s^\frown 0}$ and $\varphi_{s^\frown 1}$ have smaller Morley rank or Morley degree. $\square$

Definition 3.4. For types p we put

$$\mathrm{MR}(p) := \min\{\mathrm{MR}(\varphi) | \varphi \in p\},$$
$$\mathrm{Mdeg}(p) := \min\{\mathrm{Mdeg}(\varphi) | \mathrm{MR}(\varphi) = \mathrm{MR}(p), \varphi \in p\}.$$

Thus $\mathrm{MR}(\varphi) = \max\{\mathrm{MR}(p) | \varphi \in p\}$.

If G is a totally transcendental group, a formula $\varphi(x)$ and type $p(x)$ are called **generic** iff $\mathrm{MR}(\varphi) = \mathrm{MR}(p) = \mathrm{MR}(G) := \mathrm{MR}(\ulcorner x = x \urcorner)$.

We will need that in stable theories all types $p \in S(B)$, $B \subseteq \mathcal{M}$, $\mathcal{M} \models T$ are **definable**. First we do this for **φ-types**: We set $p \in S_\varphi(B)$ iff p is consistent and for every $\overline{b} \in B$ we have $\varphi(\overline{x}, \overline{b}) \in p$ or $\neg\varphi(\overline{x}, \overline{b}) \in p$.

Definition 3.5.

- A type $p \in S_n(B)$ is **definable** over C iff for each $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$, there is an $\mathcal{L}(C)$-formula $\psi(y)$ such that for all $\overline{b} \in B$ we have $\varphi(\overline{x}, \overline{b}) \in p \iff \mathcal{M} \models \psi(\overline{b})$.
- $\varphi(\overline{x}, \overline{y})$ is called **stable** iff for some infinite cardinal λ we have $|S_\varphi(B)| \leqslant \lambda$ for all $|B| \leqslant \lambda$.
- $\varphi(\overline{x}, \overline{y})$ has the **order property (OP)** iff there are tuples $\overline{a_i}, \overline{b_i}$, $i < \omega$ such that $\mathcal{M} \models \varphi(a_i, b_j) \iff i < j$.
- $\varphi(x, y)$ has the **binary tree property** iff there is a binary tree $(b_s)_{s \in {}^{<\omega}2}$ of parameters such that for all $\sigma \in {}^{\omega}2$ the set

  $$\{\varphi^{\sigma(n)}(\overline{x}, b_{\sigma|_n} | n < \omega\}$$

  is consistent, where $\varphi^0 := \neg\varphi$ and $\varphi^1 := \varphi$.

Theorem 3.6. The following are equivalent:

(i) φ is stable.

(ii) $|S_\varphi(B)| \leqslant |B|$ for all infinite B.

(iii) φ doesn't have (OP).

(iv) φ doesn't have the binary tree property.

(v) Every φ-type $p \in S_\varphi(B)$ is definable over B.

For the proof we need some preparation:

Lemma 3.7. If $\varphi(\overline{x}, \overline{y})$ has (OP) and $(I, <)$ is a linear order, then there are a_i, b_i, $i \in I$ such that $\models \varphi(a_i, b_j)$ iff $i < j$.

Proof. Cf. Sheet 5, Exercise 3 (B.5.3). ☐

Corollary 3.8. If $\varphi(\overline{x}, \overline{y})$ has (OP), then three are a_i, b_i, $i < \omega$ such that $\models \varphi(a_i, b_j)$ iff $i > j$.

We also need

Theorem 3.9 (Ramsey)**.** Let A be infinite, $n < \omega$, $C_1 \sqcup \ldots \sqcup C_k = [A]^n$ a colouring of the n-element subsets of A. Then there exists some infinite $A_0 \subseteq A$, $i \leqslant k$ such that $[A_0]^n \subseteq C_i$.

Proof. We use induction on n. The statement is trivial for $n = 1$. Assume that we have shown the theorem for some n. Consider a coloring c on $[A]^{n+1}$. Fix some $a_0 \in A$. We obtain a coloring on $[A \setminus \{a_0\}]^n$ as follows: For $[a_0] \cup$

$X \in [A]^{n+1}$ put $c_{a_0}(X) := c(X \cup \{a_0\})$. By the induction hypothesis, there is a monochromatic set $B_1 \subseteq A \backslash \{a_0\}$. Take $a_1 \in B_1$. Color $[B_1 \backslash \{a_1\}]^n$ by $c_{a_1}(X) := c(X \cup \{a_0\})$. Iterating this construction we obtain a chain

$$A = B_0 \supseteq B_1 \supseteq B_1 \supseteq \ldots$$

and $a_i \in B_i \backslash B_{i+1}$ such that $C(\{a_{i_0}, a_{i_1}, \ldots, a_{i_n}\})$ depends only on i_0 for all $i_0 < i_1 < \ldots < i_n$. By induction hypothesis for $n = 1$, there are infinitely many i_0 yielding the same coloring. Let A_0 be the set of such i_0. □

Theorem 3.10 (Erdős-Makkai)**.** If B is infinite and $\mathcal{S} \subseteq \mathcal{P}(B)$ such that $|B| < |\mathcal{S}|$, then there is $\langle b_i | i < \omega \rangle$, $b_i \in B$, $\langle S_i | i < \omega \rangle$, $S_i \in \mathcal{S}$ such that either

(i) $b_i \in S_j \iff j < i$ or

(ii) $b_i \in S_j \iff i < j$.

Proof. We say that X *separates* A from B if $A \subseteq X$ and $X \cap B = \varnothing$. Construct $\mathcal{S}' \subseteq \mathcal{S}$, $|\mathcal{S}'| = |B|$ such that any pair of finite subsets of B that can be separated in $\mathcal{S}$ are separated in $\mathcal{S}'$: For any two finite subsets of B put a corresponding $B_0 \subseteq B$ into $\mathcal{S}'$.

Since $|\mathcal{P}_{\text{fin}}(B)| = |B|$, we have $|\mathcal{S}'| = |B|$. Since $|\mathcal{S}'| < |\mathcal{S}|$ there is $S^* \in \mathcal{S}$ which is *not* a boolean combination of sets in $\mathcal{S}'$. We now construct sequences

$$\begin{aligned} &\langle b'_i | i < \omega \rangle \text{ in } S^*, \\ &\langle b''_i | i < \omega \rangle \text{ in } B \backslash S^*, \\ &\langle S_i | i < \omega \rangle \text{ in } \mathcal{S}', \end{aligned}$$

such that

- $\{b'_0, \ldots, b'_n\} \subseteq S_n$, $\{b''_0, \ldots, b''_n\} \subseteq B \backslash S_n$ and
- $b'_n \in S_i \iff b''_n \in S_i$ for all $i < n$.

Assume we have defined those for $i < n$. Since S^* is not a boolean combination of $S_i, i < n$, there exist $b'_n \in S^*$, $b''_n \in B \backslash S^*$ such that for all $i < n$, $b'_n \in S_i \iff b''_n \in S_i$. Let $S_n \in \mathcal{S}'$ separate $\{b'_0, \ldots, b'_n\}$ from $\{b''_0, \ldots, b''_n\}$ (this exists, since $S^* \in \mathcal{S}$ separates them).

We may assume $b'_n \in S_i$ or $b'_n \notin S_i$ for all $i < n < \omega$: Set $c(\{n, m\}) := [b_{\max(n,m)} \in S_{\min(n,m)}]$. Ramsey's Theorem (3.9) yields $N \subseteq \omega$ infinite such that $[N]^2$ is monochromatic.

In the first case put $b_i := b''_i$ ($\notin S^*$, $\rightsquigarrow$ (i)). Otherwise put $b_i := b'_{i+1}$ ($\rightsquigarrow$ (ii)). By construction we have $i \leqslant n \implies b'_i \in S_n$, $b''_i \notin S_n$. If $b'_n \in S_i$ for $i < n$, then also $b''_n \in S_i$. Hence $i < n$ iff $b_n \in S_i$ by choice of S_n. The other case is similar. □

Proof of Theorem 3.6. Clearly (ii) $\implies$ (i), (v) $\implies$ (ii).

(i) $\implies$ (iv) Suppose that φ is λ-stable and μ minimal such that $2^\mu > \lambda$. The tree $T = {}^{<\mu}2$ has cardinality $\leqslant \lambda$. If $\varphi(\overline{x}, \overline{y})$ has the binary tree property, then by the Compactness Theorem (A.14) we find $(b_s)_{s \in T}$ such that for $\sigma \in {}^{\mu}2$ the type

$$q_\sigma := \{\varphi^{\sigma(x)}(\overline{x}, b_{\sigma|_\alpha}) | \alpha < \mu\}$$

is consistent. Hence the q_σ extend to a family of pairwise distinct φ-types over $B = \{b_s | s \in T\}$, so $|B| \leqslant \lambda < 2^\mu \leqslant |S_\varphi(B)|$. ↯

(iv) $\implies$ (iii) Choose a linear ordering on $I = {}^{\leqslant\omega}2$ such that $\sigma < \sigma|_n \iff \sigma(n) = 1$ for all $\sigma \in {}^{\omega}2, n < \omega$. If $\varphi(x, y)$ has (OP), by Lemma 3.7 we find $(a_i, b_i)_{i \in I}$ such that $\models \varphi(a_i, b_j) \iff i < j$.

Thus the tree $\varphi(x, b_s)$, $s \in {}^{<\omega}2$ has the binary tree property.

(iii) $\implies$ (ii) Let $|B| \geqslant |T|$, $|S_\varphi(B)| > |B|$. The φ-type of a over B is determined by

$$S_a = \{\overline{b} \subseteq B | \models \varphi(a, \overline{b})\} \subseteq B^n.$$

Since $|B^n| = |B|$ we may assume $n = 1$. Applying Theorem 3.10 to B and $\mathcal{S} = \{S_a | a \in M\}$ we obtain $(b_i)_{i<\omega}$,$(a_i)_{i<\omega}$, $b_i \in B, a_i \in M$ such that either

- $b_i \in S_{a_j} \iff j < i$ for all $i, j < \omega$ or
- $b_i \in S_{a_j} \iff i < j$ for all $i, j < \omega$.

Thus φ has (OP).

(v) $\implies$ (iv) Suppose $\varphi(x, y)$ doesn't have the binary tree property. For a formula $\theta(x)$ let $d_\varphi(\theta)$ be the maximal n such that there is a binary three $(b_s)_{s \in {}^{<n}2}$ such that

$$\{\theta(x)\} \cup \{\varphi^{\sigma(i)}(x, b_{\sigma|_i} | i < n\}$$

is consistent for all $\sigma \in {}^{n}2$. Let $p \in S_\varphi(B)$ and let θ be a conjunction of formulae in p such that $n := d_\varphi(\theta)$ is minimal. Then

$$\varphi(x, b) \in p \iff d_\varphi(\theta(x) \wedge \varphi(x, b)) = n.$$

Note that the right hand side is definable. □

[Lecture 08, 2024-05-13][7]

Corollary 3.11 (Separation of Variables)**.** Let T be stable, $\mathcal{M} \models T$ and $A = \varphi(\mathcal{M})$ a $\varnothing$-definable subset. Then every $\mathcal{L}(\mathcal{M})$-definable subset of A is A-definable.

In other words, for every formula ψ and $\overline{c} \subseteq \mathcal{M}$ such that $\psi(M, \overline{c}) \subseteq \varphi(\mathcal{M})$,

[7] The proof of Theorem 3.6 was finished in this lecture.

there exist ψ' and $\overline{a} \subseteq A$ such that

$$\psi'(\mathcal{M}, \overline{a}) = \psi(M, \overline{c}).$$

Proof. The type $p(y) = \operatorname{tp}(\overline{c}/\varphi(\mathcal{M}))$ is definable over A by Theorem 3.6, i.e.

$$\psi(\mathcal{M}, \overline{c}) = \{a \in A | \psi(a, y) \in p\}$$

is A-definable. □

Example 3.12. This does not hold without stability: Let Γ be the random graph Consider (Γ, a) for some $a \in \Gamma$. Then $A = \Gamma_1(a) := \{b \in \Gamma : \operatorname{dist}(a, b) = 1\}$ is $\emptyset$-definable. However for $b \in \Gamma \backslash A$ the set $\Gamma_1(a) \cap \Gamma_1(b)$ is not definable over A. (Note that (Γ, a) has QE.)

[Lecture 09, 2024-05-16] If p is a type with Morley rank, then by definition there exists $\varphi \in p$ such that $\operatorname{MR}(\varphi) = \operatorname{MR}(p)$, $\operatorname{Mdeg}(\varphi) = \operatorname{Mdeg}(p)$. We call such a φ the **characterising formula**[8] for p. Then for a formula ψ we have $\psi \in p$ iff $\operatorname{MR}(\varphi \wedge \neg\psi) < \operatorname{MR}(\varphi)$,

Corollary 3.13 (MR is definable)**.** For $\psi(x, y)$ the set

$$B_{\psi,p} := \{b | \operatorname{MR}(\varphi(x) \wedge \neg\psi(x, b)) < \operatorname{MR}(\varphi)\}$$

is definable.

If $p \in S(B)$, the set $B_{\psi,p}$ is defined by an $\mathcal{L}(B)$-formula and we can evaluate this formula on arbitrary elements. If $C \supseteq B$, this defines a φ-type $q \in S(C)$, $q \supseteq p$ such that

$$\psi(x, c) \in q \iff \models \operatorname{def}_p(\psi)(c)$$

where $\operatorname{def}_p(\psi)$ is the formula defining $B_{\psi,p}$. By construction $\operatorname{MR}(q) = \operatorname{MR}(p)$.[9]

Definition 3.14. Let $A \subseteq B$, $p \in S(A)$ with $\operatorname{MR}(p) = \alpha$. Then $q \in S(B)$ with $q \supseteq p$ and $\operatorname{MR}(p) = \operatorname{MR}(q)$ is called a **non-forking extension** of p.

Remark 3.14.15 (Heir property)**.** All formulae in a non-forking extension q are (possibly with different parameters) already in p.

Lemma 3.15. Every type $p \in S(A)$ with $\operatorname{MR}(p) = \alpha$ has a non-forking extension to any set $B \supseteq A$. There are at most $\operatorname{Mdeg}(p)$ many non-forking

[8] This is not official notation.
[9] In a sense, q adds no additional information.

extensions to B and

$$\mathrm{Mdeg}(p) = \sum\{\mathrm{Mdeg}(q) | q \in S(B) \text{ non-forking extension of } p\}.$$

Proof. If φ is a characterising formula for p, the non-forking extensions of p to B are give exactly by those $\psi \in \mathcal{L}(B)$ that preserve the Morley rank (but maybe have smaller degree). □

Definition 3.16. A type with Morley rank is **stationary** iff $\mathrm{Mdeg}(p) = 1$, i.e. iff it has a unique non-forking extension to any superset of its domain.

Corollary 3.17. If $p \in S(A)$ is stationary, $B \supseteq A$ and $q \in S(B)$ a non-forking extension, then q is definable over A.

Proof. The characterising formula for p is also characterising for q. □

Remark 3.17.16. If T is totally transcendental and $\mathcal{M} \models T$ is ω-saturated, then MR and Mdeg for $\varphi \in \mathcal{L}(\mathcal{M})$ can be computed in $\mathcal{M}$. I.e. "very saturated" as in Definition 3.1 is just ω-saturated if T is totally transcendental. In particular, all types in $S(\mathcal{M})$ are stationary.

[Lecture 10, 2024-05-27]

Notation 3.17.17. Let T be totally transcendental, $\mathcal{M} \models T$, $\overline{a}, \overline{b}, A \subseteq \mathcal{M}$. We write

$$\mathrm{MR}(\overline{a}/A) := \mathrm{MR}(\mathrm{tp}(\overline{a}/A))$$

and $\overline{a} \underset{A}{\downarrow} \overline{b}$ ("$\overline{a}$ and $\overline{b}$ are **independent** over A") iff

$$\mathrm{MR}(\overline{a}/A\overline{b}) = \mathrm{MR}(a/A),$$

i.e. iff $\mathrm{tp}(\overline{a}/A\overline{b})$ is definable over A. Similarly we define $\overline{a} \underset{A}{\downarrow} B$.

$\overline{a} \underset{\varnothing}{\downarrow} \overline{b}$ is abbreviated as $\overline{a} \downarrow \overline{b}$.

Example 3.18. Let T_{tree} be the theory of cycle-free graphs such that every vertex has infinite valency. T_{tree} is a complete theory with QE in the language $d_n(x, y) :=$ "$\mathrm{dist}(x, y) = n$". Furthermore it is ω-stable (follows from QE).

Let b be a vertex. We claim that $p(x, b) = \{d_1(x, b)\}$ is a complete type.

Exercise: Work out the details

For $A \neq \varnothing$ we have $a \underset{A}{\downarrow} b$ iff the shortest path from a to b passes through $\operatorname{conv}(A)$, the convex closure of A.

For $A = \varnothing$ it is $a \underset{A}{\downarrow} b$ iff a and b belong to different connected components.

Note that $\mathrm{MR}(a/A) = \mathrm{MR}(a|\operatorname{conv}(A))$.

Theorem 3.19. If T is totally transcendental, then $a \underset{A}{\downarrow} b \iff b \underset{A}{\downarrow} a$.

Proof. Let $\mathcal{M} \models T$ be an ω-saturated model (a submodel of the monster-model) and $A \subseteq \mathcal{M}$. Wlog.

(i) $a \underset{A}{\downarrow} \mathcal{M}$ and

(ii) $b \underset{Aa}{\downarrow} \mathcal{M}$

by Lemma 3.15.

Suppose that $a \underset{A}{\not\downarrow} b$. Then by (i) $a \underset{\mathcal{M}}{\not\downarrow} b$ as $\mathrm{MR}(a/\mathcal{M}) \overset{(i)}{=} \mathrm{MR}(a/A) \overset{a \underset{A}{\not\downarrow} b}{>} \mathrm{MR}(a/Ab) \geqslant \mathrm{MR}(a/\mathcal{M})$.

We need to show that $b \underset{A}{\not\downarrow} a$.

Claim 1. $b \underset{\mathcal{M}}{\not\downarrow} a$.

Subproof. Let $\alpha := \mathrm{MR}(a/\mathcal{M})$, $\beta := \mathrm{MR}(b/\mathcal{M})$, and $\varphi(x) \in \operatorname{tp}(a/\mathcal{M})$, $\psi(y) \in \operatorname{tp}(b/\mathcal{M})$ the characteristic formulas.

Since $a \underset{\mathcal{M}}{\not\downarrow} b$, there is an $\mathcal{L}(\mathcal{M})$ formula $\chi(x,y)$, such that $\models \chi(a,b)$ (i.e. $\chi(x,b) \in \operatorname{tp}(a/b\mathcal{M})$) and $\mathrm{MR}(\chi(x,b)) < \alpha$. Wlog. $\models \varphi(x,y) \to \varphi(x) \wedge \psi(y)$.

■

If $b \underset{A}{\downarrow} a$, then by (ii), $b \underset{A}{\downarrow} \mathcal{M}$, hence $b \underset{\mathcal{M}}{\downarrow} a$ ↯.

By Corollary 3.13 the set $\{c | \mathrm{MR}(\chi(x,c)) < \alpha\}$ is $\mathcal{M}$-definable. Hence wlog. $\mathrm{MR}(\chi(x,c)) < \alpha$ for all $c \in \mathcal{M}$. Since $\mathrm{MR}(a/\mathcal{M}) = \alpha$, $\chi(a,y) \in \operatorname{tp}(b/a\mathcal{M})$ is not realized in $\mathcal{M}$. Hence $\mathrm{MR}(\chi(a,y)) < \mathrm{MR}(\psi(y)) = \beta$. □

Remark 3.19.18. Prof. Tent sometimes uses RM (french) instead of MR.

Definition 3.20. If G is a totally transcendental group, a formula $\varphi \in \mathcal{L}(G)$ and a type $p \in S_1(G)$ are **generic** if

$$\mathrm{MR}(\varphi) = \mathrm{MR}(p) = \mathrm{MR}(G) = \mathrm{MR}(\ulcorner x = x \urcorner).$$

Note that G acts by left-multiplication on the compact space $S_1(G)$: If $p \in S_1(G)$ then $p = \mathrm{tp}(x/G)$ for some x in $\tilde{G} \succ G$, then for $a \in G$, we let $ap := \mathrm{tp}(ax/G)$.

Since multiplication is a definable bijection on $\tilde{G}$, it preserves MR and Mdeg. Furthermore, the action of G on $S_1(G)$ is definable: If $\varphi(x, b) \in p$ is the characteristic formula, then $\varphi(a^{-1}x, b)$ is the characteristic formula of ap. So

$$\begin{aligned}\mathrm{Stab}(p) &= \{a \in G | ap = p\}\\ &= \{a \in G | \varphi(a^{-1}x, b) \in p\}\\ &= \{a \in G | \mathrm{MR}(\varphi(x, b) \wedge \neg\varphi(a^{-1}, b)) < \mathrm{MR}(\varphi(x, b))\}\end{aligned}$$

is a definable subgroup of G.

There are only finitely many generic types, so if $p \in S_1(G)$ is generic, then $\mathrm{Stab}(p)$ has finite index in G.

Lemma 3.21. The number of generic types in G is equal to $|G/G^0| = \mathrm{Mdeg}(G)$.

Proof. If p is generic, then $\mathrm{Stab}(p) \geqslant G^0$ and $p \in S_1(G)$ has Morley degree 1, so it has to specify in which coset of $\tilde{G}/\tilde{G}^0$ the realization lies. □

Lemma 3.22. p is generic iff $\mathrm{Stab}(p) = G^0$.

Proof. "$\Longrightarrow$" was done in Lemma 3.21.

"$\Longleftarrow$" Let $\mathrm{Stab}(p) = G^0$, $p(x) = \mathrm{tp}(a/G)$ for some $a \in \tilde{G}$ and $v \in \tilde{G}^0$ generic over G such that $b \mathop{\smile\!\!\!\!\!\!|}_{G} a$.

Then $\mathrm{tp}(a/G) = \mathrm{tp}(b \cdot a/G)$. Furthermore

$$\mathrm{MR}(b/G) = \mathrm{MR}(b/Ga) = \mathrm{MR}(ba/Ga) \leqslant \mathrm{MR}(ba/G) \overset{b \in \mathrm{Stab}(p)}{=} \mathrm{MR}(a/G).$$

Since $\mathrm{MR}(b/G)$ is maximal we have equality, i.e. a is generic over G. □

Remark 3.22.19. If $g \in G$ is generic and $a \mathop{\smile\!\!\!\!\!\!|} g$, then $a \cdot g$ is also generic:

We have $\mathrm{MR}(g) = \mathrm{MR}(g/a) = \mathrm{MR}(a \cdot g/a) \leqslant \mathrm{MR}(a \cdot g)$.

Lemma 3.23. (i) Every $g \in G$ is a product of two generics.[a]

(ii) If $A, B \subseteq G$ are generic such that $G \setminus A$ and $G \setminus B$ are not generic, then $G = A \cdot B$.

[a] This happens for many concepts of "large such that the complement is small".

Proof. (i) Let $x \mathop{\smile\!\!\!\!\!\!\mid} g$ such that x is generic. Then x^{-1} and gx^{-1} are generic and

$g = (gx^{-1}) \cdot x$.

(ii) We have $A(x), B(x) \in p(x)$ for every generic type $p(x)$.

□

Definition 3.24. A definable set $A \subseteq G$ is called **indecomposable** iff for every definable subgroup $H \leqslant G$

$$|\{aH | a \in A\}| \in \{1, \infty\},$$

i.e. either A is contained in a coset of H or it intersects infinitely many cosets.

Remark 3.24.20. A definable subgroup is indecomposable iff it is connected.

[Lecture 11, 2024-06-03]

Theorem 3.25. If G is totally transcendental, every definable subset is a disjoint union of finitely many indecomposable subsets, its **irreducible components** .

The decomposition is unique if the components are maximal.

Proof. If A is decomposable, there exists a definable subgroup $H \leqslant G$ such that $1 < n := |\{aH | a \in H\}| < \omega$.

Write $A = A_1 \sqcup \ldots \sqcup A_n$, $A_i = A \cap a_i H$. If A_i is decomposable, write $A_i = A_{i_1} \sqcup \ldots \sqcup A_{i_k}$ and so on. We obtain a finitely branching tree of finite height (the tree is finite since G is totally transcendental). The leaves of the tree form a decomposition into disjoints indecomposable sets $B_i, i \leqslant l$.

If $B \subseteq A$ is indecomposable let

$$C := \bigcup \{B_i | B_i \cap B \neq \varnothing, B_i \text{ indecomposable}\}.$$

C is indecomposable. Thus we an replace the $B_i \subseteq C$ by C and get a unique decomposition as required. □

Definition 3.26. We call S a **definable group of automorphisms** of a group G if S and the action $S \times G \to G, (s, g) \mapsto s(g)$ are definable.

Remark 3.26.21. Not every definable automorphism is contained in a definable group of automorphisms.

Theorem 3.27. Let G be stable with a definable group $S \subseteq \mathrm{Aut}(G)$ of automorphisms. Let $A \subseteq G$ be definable and S-invariant. Then A is indecomposable iff the condition from Definition 3.24 holds for all S-invariant subgroups.

Proof. Let $H \leqslant G$ be definable such that $|A/H| =: n > 1$. For $s \in S$, we have $|A/H^s| = n$,

$$K = \bigcap_{s \in S} H^s \overset{\text{Baldwin-Saxl}}{=} H^{s_1} \cap \ldots \cap H^{s_m}$$

is S-invariant and $n \leqslant |A/K| \leqslant n \cdot m$. □

Theorem 3.28 (Zilber's Indecomposability Theorem)**.** Let G be a group of finite Morley rank and let $A_i, i \in I$ be indecomposable such that $1 \in A_i$ for all $i \in I$.

Then $H := \langle A_i | i \in I \rangle$ is definable and connected and there exist $A_1, \ldots, A_m$, $m \leqslant \mathrm{MR}(H)$ such that $H = (A_1 \ldots A_m)^2$.

Proof. Since G has finite Morley rank, we find $B = A_1 \cdot \ldots \cdot A_m$ of maximal Morley rank, i.e. $\mathrm{MR}(B) = \mathrm{MR}(A_i B)$ for all $i \in I$.

Let p be a generic type, i.e. a type of maximal Morley rank in B, and let $H := \mathrm{Stab}(p)$. Then H is definable and if H divides some A_i into infinitely many cosets, then $A_i B$ contains infinitely many translates of p, which are pairwise disjoint. Then $\mathrm{MR}(A_i B) > \mathrm{MR}(B)\lightning$. Hence $A_i \subseteq H$ (since $1 \in A_i$), thus $B \subseteq H$, and we obtain $\ulcorner x \in H \urcorner \in p(x)$. Thus p is the unique generic in H, $B \subseteq H$ is generic and by Lemma 3.23 $H = B^2$. □

Theorem 3.29. Let G be a group of finite Morley rank and $H < G$ definable. Then $\mathrm{MR}(G) \geqslant \mathrm{MR}(H) + \mathrm{MR}(G/H)$.

Proof. Cf. Sheet 4, Exercise 3 (B.4.3).

The statement is clear if $|G : H|$ is finite. Otherwise $\pi\colon G \to G/H$ is interpretable, hence for a definable $A \subseteq G/H$, $\pi^{-1}(A)$ is definable. By induction we get

$$\mathrm{MR}(\pi^{-1}(A)) \geqslant \mathrm{MR}(H) + \mathrm{MR}(A).$$

□

Remark 3.29.22 (Additivity of Morley rank in groups of finite Morley rank)**.** In fact equality holds. This is however hard to prove.

Lemma 3.30. If $b \in \operatorname{acl}(aA)$, then

$$\operatorname{MR}(b/A) \leqslant \operatorname{MR}(a/A) = \operatorname{MR}(ba/A).$$

Proof. Cf. Sheet 4, Exercise 3 (B.4.3). □

4 Fields

Goal. *ω-stable fields are algebraically closed. (*Macintyre*)*

This also holdes for ω-stable integral domains of finite MR (Cherlin).

Theorem 4.1 (Macintyre)**.** If $(K, +, \cdot, \ldots)$ is an infinite ω-stable field, then K is algebraically closed.

We need two ingredients from Galois theory:

(a) (Kummer) If L/K is a cyclic Galois extension of degree n, and $\operatorname{char} K \nmid n$ and K contains all nth roots of unity, then the minimal polynomial of L/K is of the form $X^n - a$ for some $a \in K$.

    An extension of this form is called **Kummer-extension**.

(b) (Artin-Schreier) Let $\operatorname{char} K = p > 0$, L/K a Galois extension of degree p, then the minimal polynomial of L/K is of the form $X^p - X - a$ for some $a \in K$.

    An extension of this form is called **Artin-Schreier-extension**.

Remark 4.1.23. If F/K is a finite extension, then $(F, +) \cong (K^m, +)$ for $m = [F : K]$, since F is a finite dimensional K-vector space.

Choose a basis $B = \{b_1, \ldots, b_m\}$ for F/K. Then the multiplication on F is definable by $b_i \cdot b_j$. Hence $(F, +, \cdot)$ is interpretable in $(K, +, \cdot)$ using parameters from B.

[Lecture 12, 2024-06-06]

Proof of Theorem 4.1.

Claim 4.1.1. *$(K, +)$ is connected.*

Subproof. Let K^0 be the connected component of $(K, +)$.

For $a \in K \backslash \{0\}$, $x \mapsto a \cdot x$ is a definable group automorphism of $(K, +)$, hence it leaves K^0 invariant. Hence K^0 is an ideal in K, hence $K = K^0$ since K is a field. ■

From Lemma 3.21 it follows that $(K, +, \cdot)$ has a unique generic type.

Claim 4.1.2. *$K^* = (K^*)^n$, i.e. K contains n-th roots for all n.*

Subproof. If $a \notin K$ is generic over K, then a and a^n are interalgebraic over K, i.e. $a^n \in \operatorname{acl}(aK)$ and $a \in \operatorname{acl}(a^n K)$. Hence by Lemma 3.30 we have

$$\mathrm{MR}(a/K) = \mathrm{MR}(a^n/K).$$

Since a is generic over K, we get that a^n is generic as well.

Therefore $(K^*)^n \leqslant K^*$ is a generic subgroup of $(K^*, \cdot)$ and by connectedness we get equality.

■

So every element in K has n-th roots for all n. In particular, K is perfect (cf. Fact A.17.39).

If $\mathrm{char}(K) = p > 0$, then $X \mapsto X^p - X$ is a homomorphism of the additive group. So if a is generic, then so is $a^p - a$. Therefore the image is all of K, so in other words K has no Artin-Schreier extensions.

Claim 4.1.3. *If K is an infinite ω-stable field containing all m-th roots of unity for all $m \leqslant n$, then K has no Galois extensions of degree n.*

Subproof. Suppose L/K is a counter example where n is minimal.

Let q be prime, $q \mid n$. By Cauchy's theorem[10] and the Galois correspondence, there exists an intermediary field $K \subseteq F \subsetneq L$ such that L/F is Galois of degree q. By Remark 4.1.23 F is interpretable in K and hence ω-stable. Since n was minimal, we conclude $n = q$, $F = K$.

If $\mathrm{char}(K) \neq q$, the minimal polynomial for L/K is of the form $X^q - a$ for $a \in K$ (L/K is a Kummer extension). But since $(K^*)^q = K^*$, $X^q - a$ is reducible $\lightning$.

If $\mathrm{char}(K) = q$, the minimal polynomial for L/K is of the form $X^p - X - a$ for some $a \in K$ (L/K is an Artin-Schreier extension). Since $X \mapsto X^p - X$ is surjective, this is again reducible. ■

Claim 4.1.4. *If K is an infinite ω-stable field, then K contains all roots of unity.*

[10] If G is a group and $p \mid |G|$, then G contains an element of order p.

Subproof. Let n be minimal such that K doesn't contain all nth roots of unity. Let ξ be a primitive nth root of unity. Then $K(\xi)$ is a Galois extension of degree $\leqslant n-1$, contradicting Claim 4.1.3.

■

Thus K contains all nth roots of unity for all n, so by Claim 4.1.3 K has no Galois extensions. Since K is perfect, it follows that K is algebraically closed. □

Corollary 4.2. A field K of finite MR has no definable infinite proper subrings.

Proof. By Corollary 1.7 any definable subring $k \subseteq K$ is itself a field and hence algebraically closed. Since k has no algebraic extensions, either either $k = K$ or $[K : k] = \infty$. If $k \neq K$, then for any $n < \omega$, the k-vector space $k^n \subseteq K$ is definable and has $\mathrm{MR}(k^n) \geqslant n\,\mathrm{MR}(k)$ by Theorem 3.29. But $\mathrm{MR}(K)$ is finite by assumption. □

Corollary 4.3. If K is an infinite field of finite MR and $\operatorname{char} K = 0$, then K has no proper definable additive subgroups.

Any definable homomorphism $(K^n, +) \to (K^m, +)$ is K-linear. In particular, the group of definable endomorphisms of $(K, +)$ is isomorphic to $(K^*, \cdot)$.

Proof. Let $A \subseteq K$ be a definable additive subgroup and $H = \{a \in K | aA \subseteq A\}$. Then H is a definable infinite subring of K, hence $H = K$ by Corollary 4.2. Thus $A = \{0\}$ or $A = K$.

If $s\colon (K^n, +) \to (K^m, +)$ is a definable homomorphism, then the centralizer of s,

$$H = \{a \in K | \forall x \in K\, s(ax) = as(x)\}$$

is a definable infinite subring of K, hence $H = K$, so s is K-linear. □

Remark 4.3.24. There are fields of finite Morley rank with a definable subgroup $H \leqslant K^*$.

Remark 4.3.25. If K is an infinite field of finite Morley rank, $\operatorname{char} K = p > 0$ and $k := \tilde{\mathbb{F}}_p$, then every definable automorphism s of K is determined by its action on k:

If s, s' are definable automorphisms of K such that $s|_k = s'|_k$, then $\mathrm{Fix}(s's^{-1})$ is a definable subfield of K containing k, hence $\mathrm{Fix}(s's^{-1}) = K$, i.e. $s = s'$.

Therefore the group of definable automorphisms[a] of K is contained in $\mathrm{Gal}(k/\mathbb{F}_p) = \mathrm{Aut}(k) \cong \hat{\mathbb{Z}}$ (cf. Fact A.18.43).

[a]Note that in general this group is not definable.

Lemma 4.4. If K is an infinite field of finite MR, $\operatorname{char} K = p > 0$, then every definable automorphism of K is of the form Frob_p^n for some n. In particular, it is $\varnothing$-definable.

Proof. Again consider $k := \tilde{\mathbb{F}_p} \subseteq K$. Let f be an automorphism defined by $\varphi(x, y, \overline{a})$. By the Compactness Theorem (A.14) applied to Remark 4.3.25, there is some n such that $g|_{\mathbb{F}_{p^n}} = f|_{\mathbb{F}_{p^n}} \implies f = g$.

Since the automorphisms of $\mathbb{F}_{p^n}$ are of the form Frob_p^k, the claim follows. □

Corollary 4.5. Let K be an infinite field of finite MR. Then every definable $G \leqslant \mathrm{Aut}(K)$ is trivial.

Proof. Let $s \in G$. Then $\mathrm{Fix}(s)$ is definable, hence finite if $s \neq \mathrm{id}$. Hence $\mathrm{char}(K) = p$ and $G \leqslant \hat{\mathbb{Z}}$ by Lemma 4.4. G is abelian, torsion free and has finite MR, so it is divisible by Corollary 1.24 (iii). But $\hat{\mathbb{Z}}$ has no divisible elements. □

[Lecture 13, 2024-06-10]

Conjecture 4.6 (Cherlin-Zilber)**.** Any infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field.

Remark 4.6.26. Conversely, any simple algebraic group over an algebraically closed field is definable in the field as a matrix group, hence of finite MR. In fact, biinterpretability holds, i.e. if the conjecture is true every infinite simple group of finite Morley rank interprets a field.

Remark 4.6.27. The conjecture is proved for MR $\leqslant 3$.

Problem. *Using model theory, we can only talk about* ***definably simple*** *groups (i.e. there is no* definable *normal subgroup).*

However in the context of finite Morley rank, the notions of definably simple and simple coincide:

Lemma 4.7. If G is definably simple of finite MR, such that G' is infinite[a] then G is simple and in fact **boundedly simple**, i.e. for all $a \in G\backslash\{1\}$ and $g \in G$ g can be written as a product of at most $2 \cdot \mathrm{MR}(G)$ may conjugates of a.

[a] The **derived group**, also called **commutator subgroup**, is $G' := [G, G]$.

Note that being boundedly simple can be written as a first order formula. In particular, all models of the theory of a boundedly simple group are simple. This does not hold for simple groups!

Proof. Let G be an infinite, definably simple group of finite MR.

By Theorem 3.27 every infinite set $A \subseteq G$ invariant under conjugation is indecomposable. Since G is connected (otherwise the connected component would be a definable proper normal subgroup) and $Z(G) = 1$, we have $|a^G| = |G|$ for all $a \in G \backslash \{1\}$. Hence for all $a \neq 1$, we have that $a^G \cup \{1\}$ is indecomposable. So by Zilber's Indecomposability Theorem (3.28) the claim follows. $\square$

Remark 4.7.28. A theory T is called κ-categorical, if all models of T of cardinality κ are isomorphic. For example

move this to appendix?

- $(\mathbb{Q}, <)$ is $\aleph_0$-categorical,
- $(\mathbb{C}, +)$ is κ-categorical for all $\kappa \geqslant \aleph_1$,
- $(V, +, (\lambda_x)_{x \in \mathcal{F}})$ is κ-categorical for all $\kappa > |\mathcal{F}|$.

Morley's Theorem says that for all $\kappa, \lambda \geqslant \aleph_1$, a theory is κ-categorical iff it is λ-categorical. [a]

The idea of the proof is to introduce a notion of dimension.

A set is called **strongly minimal** iff it has Morley rank 1 and Morley degree 1.

Recall that acl has the following exchange property: if $a \in \mathrm{acl}(bA) \backslash \mathrm{acl}(A)$, then $b \in \mathrm{acl}(aA)$. This exchange property also holds in strongly minimal structures. We get a notion of dimension.

Matroidal hull operator?

The Baldwin-Lachlan Theorem says that a theory is $\aleph_1$-categorical iff it is ω-stable and has not **Vaughtian pair**.

A theory T has a Vaughtian pair **(VP)** iff there are models $\mathcal{M} \neq \mathcal{N}$, $\mathcal{M} \prec \mathcal{N}$ and $\varphi \in L(\mathcal{M})$, such that $\varphi(\mathcal{M})$ is infinite and $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$.

Not having a Vaughtian pair removes the possibility of certain subsets growing unevenly. For example if models of a theory T have two infinite equivalence classes, then the cardinalities of those equivalence classes might be unrelated, and in this case T is not $\aleph_1$-categorical.

A theory T is called **almost strongly minimal** iff there is a strongly

[a] Note that this does not hold for $\aleph_0$-categorical.

minimal formula φ and a finite set B, such that for every model $\mathcal{M} \models T$,

$$\mathcal{M} \subseteq \operatorname{acl}(\varphi(\mathcal{M}) \cup B).$$

Example 4.8. Consider $\mathbb{P}^2(K)$, the projective plane over an algebraically closed field K. Everything can be defined from a line (strongly minimal) and two additional points not belonging to this line.

If T is almost strongly minimal, then it is $\aleph_1$-categorical.

Proposition 4.9. Every simple group of finite MR and every field of finite MR is $\aleph_1$-categorical and in fact almost strongly minimal.

Proof. Let G be a simple group of finite MR and $A \subseteq G$ infinite and definable. Wlog. A is indecomposable (otherwise replace it by an indecomposable component). Wlog. $1 \in A$, otherwise shift A. Consider $\langle A^g | g \in G\rangle \overset{\text{Zilber}}{=} G = A^{g_1} \cdot \ldots \cdot A^{g_n}$. Since this holds in every extension of G, the definable set A has to increase in an extension of G, so G has no (VP), since $\forall g.\ \exists g_1, \ldots, g_n.\ g \in A^{g_1} \cdot \ldots \cdot A^{g_n}$ can be written as a first order formula.

We also get almost strongly minimal: Suppose φ is strongly minimal, then G is contained in $\operatorname{acl}(\varphi(G), g_1, \ldots, g_n)$.

Suppose that K is a field of finite MR (i.e. it is algebraically closed and strongly minimal as a pure[11] field). Let $A := \varphi(K)$ be an infinite, definable, indecomposable with respect to $(K, +)$ set. Wlog. $0 \in A$. Then

$$K_0 := \langle gA | g \in K\rangle \overset{\text{Zilber}}{=} g_1 A + \ldots + g_n A$$

is a definable ideal, so $K_0 = K_1$, and the same argument as in the case of groups applies. ☐

Goal. *If the Cherlin-Zilber conjecture holds, then every simple group of finite* MR *must interpret an infinite field, since this is the case in algebraic groups over algebraically closed fields We want to find the field.*

Definition 4.10. Let A be an infinite, ~~abelian~~[a] group, $G \leqslant \operatorname{Aut}(A)$. Then A is **G-minimal** iff every G-invariant subgroup of A is finite.

[a] This notion is also used for non-abelian groups.

Remark 4.10.29. A linear representation $G \rightharpoondown K^n$ is called an **irreducible representation** iff K^n has no G-invariant subspace (execept $0, K^n$). In

[11] not considering additional structure

this case by Schur's lemma $\mathrm{Cen}(G)$ is a skew field containing K.

Let $s \in \mathrm{Cen}(G)$. Then $gs(K^n) = sg(K^n) = s(K^n)$ for all $g \in G$, i.e. $s(K^n) \in \{0, K^n\}$.

Theorem 4.11 (Zilber)**.** Let A be an abelian group and $M \leqslant \mathrm{Aut}(A)$ definable inside a structure of finite MR. If A is M-minimal, then there exists a there exists a definable (algebraically closed) field K and a K-vector space structure on A such that $A \cong K^+$ and $M \leqslant K^*$ (and 0 is the only M-invariant subspace).

[Lecture 14, 2024-06-13]

Proof of Theorem 4.11. By the chain condition, there exist $a_1, \ldots, a_m \in A$ such that $\mathrm{Fix}_M(\{a_1, \ldots, a_m\}) = 1$, i.e. for $m, m' \in M$ we have $m = m'$ iff $m(\overline{a}) = m'(\overline{a})$.

Since M is infinite, there exists $a = a_i$, $i \leqslant n$ such that the orbit $M \cdot a$ is infinite.

Claim 4.11.1. *$Ma \cup \{0\}$ is an indecomposable subset of A (and clearly M-invariant).*

Subproof. By Theorem 3.27 it suffices to check the criterion for M-invariant subgroups of A. But by assumption, A is M-minimal, i.e. it has only finite M-invariant subgroups, so this is trivial. ■

By Zilber's Indecomposability Theorem (3.28) we have that $\langle Ma \rangle \leqslant A$ is definable. Clearly it is M-invariant, hence

$$A = \langle Ma \rangle = \sum_{i \leqslant k} M \cdot a$$

As in the proof of Theorem 2.5, the endomorphism ring S of A generated by M is interpretable as a quotient of $(M \cup \{0\})^k$. Since M and S are commutative, for any $s \in S$, we have that $s \cdot A$ is an M-invariant S-submodule of A and thus $s \cdot A \in \{0, A\}$. Hence S is an integral domain of finite MR, hence a field by stability, hence an algebraically closed field K with $A = K^+$, $M \leqslant K^+$, since the Morley rank is finite. □

This can be generalized further:

Theorem 4.12. Let A be an abelian group, and $G \leqslant \mathrm{Aut}(A)$ definable in a structure of finite Morley rank, where G is connected and

(i) there exists an infinite, definable, abelian normal subgroup $M \trianglelefteq G$,

(ii) there exists a definable, M-invariant and M-minimal subgroup $B \leqslant A$

such that $A = \langle gB | g \in G \rangle$.

Then there exists a definable, algebraically closed field K and a finite dimensional K-vector space on A, such that G acts K-linearly on A and M acts as K-scalars, i.e. $M \leqslant K^*$, $M \leqslant Z(G)$.

Proof.

Claim 1. *M^0 acts non-trivially on B.*

Subproof. Otherwise $gM^0g^{-1} = M^0$ would act trivially on gB, hence by (ii) trivially on A. Thus $M^0 = \{\mathrm{id}\}$ (since it is a subgroup of the automorphism group), so M is finite $\lightning$. ■

So ${}^{M}\!/\!_{\mathrm{Ann}(B)}$[12] and B satisfy the assumptions of Theorem 4.11. Thus $B \cong K^+$ and ${}^{M}\!/\!_{\mathrm{Ann}(B)} \leqslant K^*$ for some algebraically closed field K.

Let R be the endomorphism ring of A generated by M. The action of R on B arises from the algebraically closed field K, so $\mathrm{Ann}_R(B) = I \subseteq R$ is a maximal ideal, i.e. $K = {}^{R}\!/\!_{I}$.

Take $g_1, \ldots, g_n \in G$ and let $B_i \coloneqq g_iB$. Since $M \trianglelefteq G$, we have $\mathrm{Ann}(B_j) = g_jIg_j^{-1} \eqqcolon I_j$.

Claim 2. *All the I_j coincide.*

Subproof. Suppose we can find $g_1, \ldots, g_n \in G$ such that the I_j are pairwise distinct. All I_j are maximal ideals, hence coprime, i.e. $I_j + I_k = R$ for all $j \neq k$.

Claim 1. *The corresponding R-modules B_j, $j \leqslant n$ form a direct sum.*

Subproof. Let $x_j \in B_j$ be such that $\sum x_j = 0$. By the Chinese Remainder Theorem there exists $s_i \in R$, such that $s_i \in 1 + I_i$ but $s \in I_j$ for all $j \neq i$. We get

$$s_i \left(\sum x_j\right) = x_i = 0,$$

hence $x_1 = x_2 = \ldots = x_r = 0$. ■

Hence $\mathrm{MR}(A) \geqslant \mathrm{MR}(B_1 + \ldots + B_n) \geqslant n \cdot \mathrm{MR}(B)$. Therefore there are only finitely many distinct ideals I_j. Let $\{I_1, \ldots, I_n\}$ be the set of all of these with $I_j = \mathrm{Ann}(B_j)$.

Consider the action of G on $\{I_1, \ldots, I_n\}$. Since the I_j are conjugate, this action is transitive.

[12] Recall that the **annihilator** is defined as $\mathrm{Ann}(B) = \{m \in M | mB = 0\}$.

Claim 2. *The action of G on $\{I_1, \ldots, I_n\}$ is definable.*

Subproof. Pick $i_{jk} \in I_j \backslash I_k$ for all $1 \leqslant j \neq k \leqslant n$. Let $g \in G$. Then $I_j^g = I_l$ iff for all $k \leqslant n$

$$\underbrace{(i_{jk})^g \in \mathrm{Ann}(B_l)}_{\text{definable}} = I_l.$$

■

Since G is connected, we get that the action is trivial (otherwise the stabilizers would be definable proper subgroups). So since the action is both trivial and transitive, there can only be on ideal. ■

Since A is generated by the gB, $g \in G$ and $\mathrm{Ann}(gB) = \mathrm{Ann}(B) = I$, we have $I \leqslant \mathrm{Ann}(A)$, i.e. $I = 0$. Thus $R = K$.

Claim 3. *The action of K on A is definable.*

Subproof. By Zilber's Indecomposability Theorem (3.28), we have $A = \sum_{i \leqslant n} g_i B$. The action of $M = {}^{M}\!/_{\mathrm{Ann}(B)}$ on A shows that every $s \in R$ can be written as

$$s = m_1 + \ldots + m_n,$$

so $K = \sum_{i \leqslant n} M$. ■

By construction, the elements of M act as scalars on A. G acts on M be conjugation, hence it induces a definable group of automorphisms on K. By Corollary 4.5 this induced group is trivial, i.e. $M \leqslant Z(G)$, hence G acts K-linearly on A. □

Corollary 4.13. Let G be a definable (in a structure of finite MR) group of automorphisms of an abelian group A, where A is G-minimal. Then either

- G has an infinite center or
- G has no definable nontrivial abelian normal subgroup (i.e. G is **definably semi-simple**).

Proof. This follows from Theorem 4.12: If $M \trianglelefteq G$ is an infinite definable abelian subgroup, then using finiteness of the Morley rank we find a definable M-invariant, M-minimal subgroup $B \leqslant A$. Since A is G-minimal and $M \trianglelefteq G$ we have $A = \sum g_i B$. By Theorem 4.12 we get $M \leqslant Z(G)$ and G acts K-linearly. □

Remark 4.13.30. We call a group G **minimal'**[a] iff it doesn't contain proper infinite definable subgroups.

[a] Confusingly, in the lecture this was called "minimal" as well. Note that in the previous definition all definable subsets were considered.

The proof of Theorem 1.3 shows:

Theorem 4.14. Minimal' groups are abelian.

Corollary 4.15. ??

(i) Any group of Morley rank 1 is virtually abelian.

(ii) Any ω-stable infinite group contains an infinite, definable, abelian subgroup.

Proof. (i) G^0 is minimal', hence abelian.

(ii) Since G has finite MR, there exists and infinite, definable subgroup $H \leqslant G$ of minimal MR. Then H is minimal', hence abelian.

□

Remark 4.15.31.

(i) If A is a minimal' group, then it is G-minimal for every definable $G \leqslant \mathrm{Aut}(A)$.

(ii) Algebraic groups of dimension 1 are abelian. Algebraic groups of dimension 2 are solvable.

A Recap

[Tutorial 01, 2024-04-16]

A.1 Groups

Definition† A.0.32. Let G be a group. The **center** of G is

$$Z(G) := \{z \in G | \forall g \in G.\ zg = gz\}.$$

Definition† A.0.33. Let G be a group and $S \subseteq G$ a subset.

The **centralizer** of S in G is

$$C_G(S) := \{g \in G | \forall s \in S.\ gs = sg\}.$$

The **normalizer** of S in G is

$$N_G(S) := \{g \in G | gS = Sg\}.$$

Clearly $C_G(S) \leqslant N_G(S) \leqslant G$ are subgroups.

Let $A, B \leqslant G$ be two subgroups. Then A is **normalized** by B (B **normalizes** A) iff for every $b \in B$, $A^b = A$, i.e. $B \leqslant N_G(A)$.

Similarly, A is **centralized** by B (B **centralizes** A) iff $B \leqslant C_G(A)$.

Definition† A.0.34. Let p be prime. A **p-group** is a group in which the order of every element is a power of p.

Let G be a group. A **p-Sylow subgroup** of G is a maximal p-subgroup of G.

Sylow theorems

Definition† A.0.35. Let G be a group. The **order** of G, $\operatorname{ord}(G)$, is the number of elements of G.

The **order** (**period length** / **period**) of $g \in G$ is the order of $\langle g \rangle$, the subgroup generated by g.

G is a **torsion group** (**periodic group**) iff every element has finite order.

The **exponent** of a torsion group G is the least common multiple of orders of the group elements, i.e. the least $n \in \mathbb{N}$ such that $\forall g \in G.\ g^n = 1$. (This does not necessarily exist.)

Definition A.1. A group G is called **simple** iff $\{1\}$ and G itself are the only normal subgroups of G.

Semidirect product

A.1.1 Group actions

Definition A.2. Let G be a group and X a set. A **group action** $G \curvearrowright X$ is a group homomorphism $\pi\colon G \to \mathrm{Sym}(X)$. For $g \in G, x \in X$ we will write $g.x := \pi(g)(x)$.

- The action is **transitive** iff $\forall x, x' \in X.\ \exists g \in G.\ g.x = x'$.
- The action is **n-transitive** iff for all pairwise distinct $x_1, \dots, x_n \in X$ and all pairwise distinct $x'_1, \dots, x'_n \in X$ there exists $g \in G$ such that $g.x_i = x'_i$ for all $i \leqslant n$.

  It is **sharply n-transitive** iff there exists exactly one such g.
- The action is **faithful** iff π is injective.
- The action is **free** iff no non-trivial element has a **fixpoint**, i.e. $\forall g \in G \backslash \{1\}, x \in X.\ g.x \neq x$.
- The action is **regular** iff it is transitive and free.
- The **stabilizer** of $x \in X$ is the subgroup $G_x := \{g \in G : g.x = x\}$.
- The **orbit** of $x \in X$ is $G.x := \{g.x | g \in G\}$.

Definition A.3. For a subgroup $H \leqslant G$ the **index** of H in G, $|G : H|$ is defined as $|\{gH / g \in G\}|$.

Theorem A.4 (Orbit stabilizer theorem)**.** Let $G \curvearrowright X$, $x \in X$. Then $|G.x| = |G : G_x|$.

Proof. The map

$$\begin{aligned} \varphi\colon G.x &\longrightarrow G/G_x \\ g.x &\longmapsto gG_x \end{aligned}$$

is bijective. □

A.1.2 Nilpotent and solvable groups

Definition A.5. Let G be a group. The **commutator** or **derived subgroup** of G, denoted $[G, G]$ or G', is defined as $\{[g, g'] : g, g' \in G\}$. $[G, G]$ is the smallest subgroup of G such that G/G' is abelian.

We recursively define $G^{(0)} := G$ and $G^{(n+1)} := [G^{(i)}, G^{(i)}]$.

We say that G is **solvable** if $G^{(n)} = 1$ for some $n \in \mathbb{N}$.

Proposition A.6. G is solvable iff we have a sequence $1 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_m = G$ such that H_{i+1}/H_i is abelian.

Example A.7. Let K be a field, $\mathrm{AGL}_1(K) := \{x \mapsto ax+b | a \neq 0, b \in K\} \cong K \rtimes K$ the group of affine transformations.

We have $\mathrm{AGL}_1(K)/K^+ \cong K^+$, so $1 \trianglelefteq K^+ \trianglelefteq \mathrm{AGL}_1(K)$.

Definition A.8. A **linear algebraic group** of a field k is a subgroup of $\mathrm{GL}_1(K)$, e.g. $\mathrm{SL}_1(K)$. For such a group, a Borel subgroup is a maximal closed connected solvable group.

Example A.9.

- A Borel subgroup for $\mathrm{GL}_n(K)$ is the group of upper triangular matrices.
- A Borel subgroup for $\mathrm{SL}_n(K)$ is the group of upper **unitriangular** matrices (i.e. upper triangular matrices with only 1 on the diagonal).

Definition A.10. Let G be a group. We define $G^{[n]}$ inductively be $G^{[0]} := G$, $G^{[n]} := [G, G^{[n-1]}]$.

We say that G is (n-step) **nilpotent** iff there exists $n \in \mathbb{N}$ (minimal) such that $G^{[n]} = 1$.

Nilpotent groups are solvable.

Proposition A.11. The following are equivalent

- G is nilpotent.
- There exists a finite ascending central series, $\zeta_0(G) = 1 \trianglelefteq \zeta_1(G) = Z(G) \trianglelefteq \ldots \trianglelefteq \zeta_n(G) = G$ with $\zeta_{i+1}(G) = \{g \in G : g\zeta_i(G) \in Z(G/\zeta_i(G))\}$, that is $\zeta_{i+1}(G)$ is the subgroup such that $\zeta_{i+1}(G)/\zeta_i(G) = Z(G/\zeta_i(G))$.

Example A.12.

- For a field K, the group of upper unitriangular matrices is nilpotent.
- The **quaternion group** $Q_8 = \langle a, b | a^4 = e, a^2 = b^2, ba = a^{-1}b \rangle$ is nilpotent.

If a group is nilpotent / solvable, then its subgroups have the same property.

A.2 Model Theory

It was recommended to read the first two chapters of [TZ12] (Structures, Languages, Theories, Elementary Substructures, Compactness Theorem, Löwenheim-Skolem) as an introduction to model theory.

A.2.1 The Compactness Theorem

Definition A.13. Let $\mathcal{L}$ be a first order languages and Σ a set of sentences in the language $\mathcal{L}$. We say that Σ is **satisfiable** iff there exists an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \sigma$ for all $\sigma \in \Sigma$ (or short $\mathcal{M} \models \Sigma$).

We say that Σ is **finitely satisfiable** iff every finite subset $\Sigma' \subseteq \Sigma$ is satisfiable.

Theorem A.14 (Compactness Theorem)**.** Σ is satisfiable iff it is finitely satisfiable.

A.3 Saturated Models

Definition A.15. Let T be a complete theory with infinite models. We say that $\mathcal{M} \models T$ is **κ-saturated** iff for $A \subseteq \mathcal{M}$, $|A| < \kappa$ every $p \in S_n(A)$ is realized in $\mathcal{M}$.

We say that $\mathcal{M}$ is **saturated** iff it is $|\mathcal{M}|$-saturated.

It is easy to construct κ-saturated models using the Compactness Theorem (A.14). However saturated models can not be constructed this easily, as the model might have too many types. (So one needs to use inacessible cardinals.)

We will often consider a **monster model**, which is very saturated, in the sense that it realizes all the types we care about and only consider submodels of this.

Definition A.16. A model $\mathcal{M} \models T$ is called **κ-homogeneous**, iff for all $A, B \subseteq \mathcal{M}$ of size at most κ and $f\colon A \to B$ is an elementary map, then for all $a \in \mathcal{M}$, f can be extended to $f\colon A \cup \{a\} \to B \cup \{b\}$ for some $b \in \mathcal{M}$.

$\mathcal{M}$ is called **homogeneous** iff it is $|\mathcal{M}|$-homogeneous.

Fact A.16.36. κ-saturated models are κ-homogeneous: Consider $\mathrm{tp}(a/A)$. Since the model is κ-saturated, it realizes the same type over B.

Theorem A.17. If $\mathcal{M}$ and $\mathcal{N}$ are saturated models of T of the same cardinality, then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

Proof. Back-and-forth. □

Fact A.17.37. Let $\mathcal{M}$ be saturated. Then $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ iff there exists an automorphism of $\mathcal{M}$ fixing A pointwise and sending a to b.

A.4 Fields

Let $K \subseteq L$ be fields. We can view L as a K-vector space. $[L : K] := \dim_K L$ is called the **degree** of the field extension. The field extension is called **algebraic** iff every element of L is a zero of a polynomial in $K[X]$.

Fact A.17.38. Finite field extensions are algebraic.

An extension L/K is **separable** iff for every $l \in L$, the minimal polynomial of l is separable, i.e. it has no multiple roots (equivalently its formal derivative does not vanish). This holds trivially in characteristic 0.

A field is called **perfect** iff all algebraic extensions of it are separable.

Fact A.17.39. If $\operatorname{char} K = p$, then K is perfect iff $\mathrm{Frob}_p \colon x \mapsto x^p$ is surjective.[a]

In particular, finite fields are perfect.

[a]Recall that morphisms of fields are always injective.

L/K is **normal** iff every irreducible polynomial in $K[X]$ that has a zero in L splits into linear factors in L.

L/K is **Galois** iff it is separable and normal. In this case let $\mathrm{Gal}(L/K) := \mathrm{Aut}_K(L)$.

Fact A.17.40. If L/K is Galois, then $|\mathrm{Gal}(L/K)| = [L : K]$.

Theorem A.18 (Fundamental Theorem of Galois Theory)**.** If L/K is Galois, there is a bijection between intermediate fields and subgroups of $\mathrm{Gal}(L/K)$, sending and intermediate field $K \subseteq M \subseteq L$ to $\mathrm{Aut}(L/K)$.

Fact A.18.41. If ξ is a primitive n-th root of unity, then the minimal polynomial of ξ has degree at most $n - 1$. (Note that $X^n - 1$ factors as $(X - 1) \cdot \ldots$)

Fact A.18.42. The algebraic closure of $\mathbb{F}_p$ is $\tilde{\mathbb{F}_p} = \bigcup_n \mathbb{F}_{p^n}$.

Fact A.18.43. $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ is a cyclic group of order n, generated by

$\langle \mathrm{Frob}_p \rangle$.

It is $\mathrm{Gal}(\tilde{\mathbb{F}_p}/\mathbb{F}_p) = \hat{\mathbb{Z}} = \varprojlim_{n<\omega} C_n$, the profinite completion of the integers.

B Exercises

B.1 Sheet 1

[Tutorial 02, 2024-04-23]

Definition B.1. The **ascending central series** of a group G is the chain $Z_0(G) \leqslant Z_1(G) \leqslant \ldots$ of subgroups of G defined inductively by $Z_0 := \{1\}$ and $Z_i(G)/Z_{i-1}(G) := Z(G/Z_{i-1}(G))$ for $i > 0$. A group G is **nilpotent** if $Z_n(G) = G$ for some $n \in \mathbb{N}$.

Definition B.2. A **Sylow-p-subgroup** of G is a p-group that is maximal wrt. inclusion.

Usually this is only defined for finite groups, where Sylow's theorem holds. However, we also consider Sylow-subgroups of infinite groups. The following version of Sylow's theorem holds:

Theorem B.3 (Sylow's theorem for infinite groups)**.** Let P be a Sylow-p-subgroup of G. If P has finitely many conjugacy classes in G, then all Sylow-p-subgroups of G are conjugate.

Definition B.4. A **characteristic subgroup** is a subgroup that is fixed by every automorphism.

This is a stronger condition than being normal (i.e. fixed under the automorphisms of the for $h \mapsto ghg^{-1}$.)

B.1.1 Exercise 1

TODO

B.1.2 Exercise 2

Let G be a group and $P \leqslant G$ be a p-Sylow subgroup of G.

(a) Show that P is a characteristic subgroup of $N_G(P)$. Deduce that $N_G(N_G(P)) = N_G(P)$.

(b) Suppose that G is nilpotent.

- Show that $N_G(P) = G$, i.e. P is normal in G.
- Deduce that G has a unique p-Sylow subgroup for each prime p.

- Conclude that any finite nilpotent group is the direct sum of its p-Sylow subgroups.

B.1.3 Exercise 3

(a) Let G be a group of exponent p,[13] where p is prime. Let $g \in G\backslash\{1\}$.

Show that no two distinct elements of $\langle g \rangle$ are conjugate and deduce that G has at least p conjugacy classes.

(b) Let G be a group. Suppose $g \in G\backslash\{1\}$ has finite order and $G = g^G \cup \{1\}$. Show that $|G| = 2$.

B.1.4 Exercise 4

Definition B.5. A transitive group action $G \curvearrowright X$, with $|X| > 1$ is called **primitive** iff each stabilizer G_x is a maximal proper subgroup of G.

(a) Let G be a group acting transitively on a set X.

Suppose that for some $x \in X$, the stabiliser G_x is normal. Show

(i) $G_y = G_x$ for every $y \in X$.

(ii) G/G_x acts regularly[14] on X.

(b) Let $G \curvearrowright X$ be primitive and G nilpotent. Then $|X|$ is prime.

B.2 Sheet 2

[Tutorial 03, 2024-04-30]

B.2.1 Exercise 1

Let G be a group considered as a structure in a language $\mathcal{L} \supseteq \mathcal{L}_{\text{group}}$.

(a) Let $\varphi(x, y_1, \ldots, y_s)$ be an $\mathcal{L}$-formula such that for any $\overline{g} \in G^s$, $H_{\overline{g}} := \varphi(G, \overline{g})$ is a finite index subgroup of G.

Then the following are equivalent:

(i) The index is uniformly bounded, i.e. $\exists n \in \mathbb{N}.\ \forall \overline{g} \in G^s.\ |G : H_{\overline{g}}| < n$.

(ii) For every model G' of $\mathrm{Th}(G)$ and every $\overline{g} \in (G')^s$, $H_{\overline{g}} = \varphi(G', \overline{g})$ is a finite index subgroup of G'.

(b) Suppose that for every $G' \equiv G$ and $g \in G'$, the conjugacy class $g^{G'}$ is finite. Then $|g^{G'}|$ is uniformly bounded.

[13] We say that G is a group of **exponent** p iff $g^p = 1$ for all $g \in G$.
[14] A group action is called **regular** iff $g \mapsto gx$ is bijective for all x.

B.2.2 Exercise 2

Let G be an infinite stable group. Show that $G\backslash\{1\}$ does not form a single conjugacy class.

B.2.3 Exercise 3

Let G be an NIP group in a language $\mathcal{L}$. Let $\varphi(x, y_1, \ldots, y_s)$ be an $\mathcal{L}$-formula such that $H_{\overline{g}} := \varphi(G, \overline{g})$ is a subgroup for every $\overline{g} \in G^s$. Let $k \in \mathbb{N}$ and let

$$X_k := \{\overline{g} \in G^s : |G : H_{\overline{g}}| \leqslant k\} \subseteq G^s.$$

Show that $\bigcap_{\overline{g} \in X_k} H_{\overline{g}}$ is a finite index subgroup of G which is $\varnothing$-definable.

B.2.4 Exercise 4

Let G be a stable group. Then any abelian subgroup of G is contained in a definable abelian subgroup of G.

B.3 Sheet 3

[Tutorial 04,]

B.3.1 Exercise 1

TODO

B.3.2 Exercise 2

Let p be prime and V and $\aleph_0$-dimensional $\mathbb{F}_p$-vector space. Then $(V, +)$ is ω-categorical.

B.3.3 Exercise 3

Let $\mathcal{M}$ be a countable infinite ω-categorical $\mathcal{L}$-structure and let $n \in \mathbb{N}$.

(a) There are only finitely many possibilities for the type of an n-tuple from $\mathcal{M}$.

(b) For each such type, the set of n-tuples form $\mathcal{M}$ with that type (i.e. the $\equiv$-equivalence class) is defined by an $\mathcal{L}$-formula.

(c) Two n-tuples $\overline{a}, \overline{b} \in \mathcal{M}^n$ have the same type iff they are in the same $\mathrm{Aut}(\mathcal{M})$-orbit.

(d) The $\mathrm{Aut}(\mathcal{M})$-invariant subsets of $\mathcal{M}^n$ are precisely the subsets definable by $\mathcal{L}$-sentences.

B.3.4 Exercise 4

TODO

B.4 Sheet 4

[Tutorial 05, 2024-05-14]

B.4.1 Exercise 1

No infinite field is ω-categorical.

B.4.2 Exercise 2

Let $\mathcal{M}$ be a structure. Let X be a definable set and $Y_i \subseteq X$ definable subsets for $i \in \omega$ with $\mathrm{MR}(Y_i) \geqslant \alpha$ but $\mathrm{MR}(Y_i \cap Y_j) < \alpha$ for $i \neq j$. Then $\mathrm{MR}(X) \geqslant \alpha + 1$.

B.4.3 Exercise 3

Let $\mathcal{M}$ be a structure, $k \in \mathbb{N}$, X a definable set and $Y_i \subseteq X$ definable subsets for $i \in \omega$ with $\mathrm{MR}(Y_i) \geqslant \alpha$. Suppose that for any $I \subseteq \omega$ with $|I| = k$, $\mathrm{MR}\left(\bigcap_{i \in I} Y_i\right) < \alpha$. Show $\mathrm{MR}(X) \geqslant \alpha + 1$.

B.5 Sheet 5

[Tutorial 06, 2024-05-28]

B.5.1 Exercise 1

Let M be a sufficiently saturated structure (as in the definition of MR). Let X, Y be non-empty definable sets and let $f \colon X \to Y$ be a definable function (i.e. the graph of f is a definable set).

(a) Suppose that f is surjective. Show that $\mathrm{MR}(X) \geqslant \mathrm{MR}(Y)$.

Furthermore, show for any $n \in \mathbb{N}$: If $\mathrm{MR}(f^{-1}(b)) \geqslant n$ for every $b \in Y$ and $\mathrm{MR}(Y) < \omega$, then $\mathrm{MR}(X) \geqslant \mathrm{MR}(Y) + n$.

(b) Suppose that $f^{-1}(b)$ is finite for every $b \in Y$. Show that $\mathrm{MR}(X) \leqslant \mathrm{MR}(Y)$.

B.5.2 Exercise 2

Let $(G, +, \dots)$ be a totally transcendental connected[15] abelian group, written additively. Let $n \in \mathbb{N}$ and suppose that the n-torsion subgroup $G[n] = \{x \in G : nx = 0\}$ is finite. Show that G is n-divisible, i.e. $nG = \{nx : x \in G\} = G$.

B.5.3 Exercise 3

Let T be an $\mathcal{L}$-theory and let $\varphi(x, y)$ be an $\mathcal{L}$-formula.

(a) Suppose that φ has the order property. Let $(I, <)$ be a linear order. Show that in some $\mathcal{M} \models T$ there are $\overline{a}_i \in \mathcal{M}^{|x|}$, $\overline{b}_i \in \mathcal{M}^{|y|}$ such that $\mathcal{M} \models \varphi(a_i, b_j)$ iff $i < j$.

(b) Suppose that φ has the binary tree property. Let μ be a cardinal. Show that in some $\mathcal{M} \models T$ there are a_σ for $\sigma \in 2^\mu$ and b_s for $s \in 2^{<\mu}$ such that $\mathcal{M} \models \varphi(a_\sigma, b_s)$ iff $s \subseteq \sigma$.

[15] i.e. it has no proper definable subgroup of finite index

B.5.4 Exercise 4

Any boolean combination of stable formulas $\varphi_i(x, y)$ is stable.

B.6 Sheet 6

[Tutorial 07, 2024-06-04]

For subsets A, B, C of a structure put $A \mathop{\downarrow}_B C$ iff $a \mathop{\downarrow}_B C$ for any finite $a \in A^{<\omega}$, i.e. for any such tuple $\mathrm{MR}(a/BC) = \mathrm{MR}(a/B)$

B.6.1 Exercise 1

The following properties of independence hold for all subsets $A, B, C \subseteq \mathcal{M}$ and finite tuples $a \in \mathcal{M}^{<\omega}$ in a model $\mathcal{M}$ of a totally transcendental theory:

(a) (Monotonicity and transitivity) $a \mathop{\downarrow}_A BC \iff \left(a \mathop{\downarrow}_A B \wedge a \mathop{\downarrow}_{AB} C\right)$.

(b) (Local character) $\exists B_0 \overset{\text{finite}}{\subseteq} B$ such that $a \mathop{\downarrow}_{B_0} B$.

(c) (Finite character) $A \mathop{\downarrow}_B C$ iff $A \mathop{\downarrow}_B c$ for every $c \in C^{<\omega}$.

(d) (Symmetry) $A \mathop{\downarrow}_B C \iff C \mathop{\downarrow}_B A$.

(e) (Existence) Suppose $\mathcal{M}$ is $|BC|^+$-saturated. Then there exists $a' \in \mathcal{M}^{|a|}$ with $\mathrm{tp}(a'/B) = \mathrm{tp}(a/B)$ and $a' \mathop{\downarrow}_B C$.

(f) (Algebraicity) $a \mathop{\downarrow}_B a \iff \mathrm{MR}(a/B) = 0$.

B.6.2 Exercise 2

Let C be a group of finite Morley rank. Suppose $H_1, H_2 \trianglelefteq G$ are definable normal subgroups with $H_1 \cap H_2 = \{e\}$. Show that $H = \langle H_1, H_2 \rangle$ is definable and $\mathrm{MR}(H) \geqslant \mathrm{MR}(H_1) + \mathrm{MR}(H_2)$.

B.6.3 Exercise 3

Let G be a group of finite Morley Rank. The **definable socle** S of G is the subgroup generated by the minimal definable non-trivial normal subgroups of G. Suppose that G has no non-trivial finite normal subgroup. Show that S is definable.

B.7 Sheet 7

[Tutorial 08, 2024-06-11]

Let T_{tree} be the theory of non-empty cycle-free graphs in which every vertex has infinite valency. We consider the models in the language with binary predicates d_n, where $d_n(x, y)$ holds iff n is the graph distance between x and y.

B.7.1 Exercise 1

(a) Any countable model $\mathcal{M}$ of T_{tree} is homogeneous, i.e. if $\overline{a}, \overline{b} \in \mathcal{M}$ have the same quantifier-free type over a finite set $A \subseteq \mathcal{M}$, then there is an automorphism of $\mathcal{M}$ fixing A pointwise and taking a to b.

(b) T_{tree} has quantifier elimination.

(c) A countable model of T_{tree} is saturated iff it has infinitely many connected components.

B.7.2 Exercise 2

(not that important)

For subsets A, B, C of a model of T_{tree}, write $A \mathop{\downarrow}_{C} B$ if any path from any element of A to any element of B includes some element of the convex hull of C.

(a) Let $\overline{xy}$ denote the unique shortest path from x to y.

- (Monotonicity and transitivity) $a \mathop{\downarrow}_{A} BC \iff \left(a \mathop{\downarrow}_{A} B \wedge a \mathop{\downarrow}_{AB} C\right)$.
- (Finite character) $A \mathop{\downarrow}_{B} C$ iff $A \mathop{\downarrow}_{B} c$ for every $c \in C^{<\omega}$.
- (Symmetry) $A \mathop{\downarrow}_{B} C \iff C \mathop{\downarrow}_{B} A$.
- (Existence) Suppose $\mathcal{M}$ is $|BC|^+$-saturated. Then there exists $a' \in \mathcal{M}^{|a|}$ with $\operatorname{tp}(a'/B) = \operatorname{tp}(a/B)$ and $a' \mathop{\downarrow}_{B} C$.

(b) (Local character) For any finite tuple a, $\exists B_0 \overset{\text{finite}}{\subseteq} \operatorname{conv}(B)$ such that $a \mathop{\downarrow}_{B_0} B$.

(c) (Stationarity over arbitrary sets) If $a \mathop{\downarrow}_{B} C$ and $a' \mathop{\downarrow}_{B} C$ and $\operatorname{tp}(a/B) = \operatorname{tp}(a'/B)$, then $\operatorname{tp}(a/BC) = \operatorname{tp}(a'/BC)$.

B.7.3 Exercise 3

Let G be a group of finite Morley rank and let X be an *infinite*[16] indecomposable subset. Show that the normal subgroup $\langle\langle X \rangle\rangle$ generated by X (the minimal normal subgroup containing X) is definable.

[16] this was wrong on the sheet

B.7.4 Exercise 4

Let K be a field of finite Morley rank, and let $X \subseteq K$ be an infinite definable subset. Show that there exists a finite sequence of elements $a_1, \ldots, a_n \in K$ such that $K = \sum_i a_i X = \{\sum_i a_i x_i | x_1, \ldots, x_n \in X\}$.

B.8 Sheet 8

[Tutorial 09, 2024-06-18]

B.8.1 Exercise 1

Let $\mathcal{M}$ be an ω-saturated model of a totally transcendental theory. Let $\mathcal{M}^* \succ \mathcal{M}$ be an elementary extension, $a, b \in (\mathcal{M}^*)^{<\omega}$ and suppose $a \underset{M}{\downarrow} b$. Let $\varphi(x, y)$ be a formula and suppose $\mathcal{M}^* \models \varphi(a, m)$.

Then there exists $m \in \mathcal{M}^{|b|}$ such that $\mathcal{M}^* \models \varphi(a, m)$.

B.8.2 Exercise 2

Let G be a connected, ω-saturated, totally transcendental group. Let $\varphi(x)$ be a generic[17] $\mathcal{L}(G)$-formula in one variable, $G^* \succ G$ an elementary extension and $a \in G^*$.

Then $G^* \models \varphi(g \cdot a)$ for some $g \in G$.

We need that every generic formula is an element of some generic type.

B.8.3 Exercise 3

Let G be an ω-saturated, totally transcendental group and $X \subseteq G$ definable. Then the following are equivalent:

(a) $\mathrm{MR}(X) = \mathrm{MR}(G)$.

(b) $\exists g_1, \ldots, g_n \in G.\ \bigcup_i g_i X = G$.

B.8.4 Exercise 4

Let K be a division ring.

(a) Suppose that the center $Z(K)$ if K is algebraically closed as a field and that $\dim_{Z(K)} K < \infty$.

Then $Z(K) = K$.

(b) Conclude from this and Macintyre's Theorem (4.1) that any division ring of finite Morley rank with infinite centre is an algebraically closed field.

B.9 Sheet 9

[Tutorial 10, 2024-06-25]

[17] i.e. $\mathrm{MR}(\varphi) = \mathrm{MR}(G)$

B.9.1 Exercise 1

(a) Let H be a connected group of finite Morley rank acting definably and transitively on a finite set $S \neq \varnothing$. Then $|S| = 1$.

(b) Let G be a group of finite Morley rank, H a connected definable subgroup and $g \in G$.

   (i) g^H is indecomposable.

   (ii) The subgroup generated by $[g, H]$ is definable.

B.9.2 Exercise 2

Let G be a group of finite Morley rank and let H and K be definable subgroups of G. Suppose that H is infinite, K-normalized (i.e. $H^k = H$ for $k \in K$) and K-minimal for the conjugation action of K. Show that H is connected.

B.9.3 Exercise 3

(a) Any totally transcendental integral domain is a field.

(b) Let K be a field of finite Morley rank. Show that K does not contain a proper, infinite, definable subring.

(c) Let K be a field of finite Morley rank in which a proper, infinite subgroup T of the multiplicative group is definable.[18]

   (i) Suppose that $T \lneq K^*$ is infinite, definable and connected. Then T is indecomposable as a definable subset of the additive group.

   (ii) Show that the additive subgroup generated by T is the whole of K.

B.9.4 Exercise 4

Let G be a connected group of Morley rank $n \in \omega$. Show that if G is solvable (nilpotent), then it is n-step solvable (nilpotent).

[18] Such fields are called **bad fields**.

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