Logic II

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These are my notes on the lecture Logic II taught by Ralf Schindler in winter 23/24 at the University Münster.

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If you find errors or want to improve something, please send me a message: `lecturenotes@jrpie.de`.

Warning 0.1. This is not an official script.

These notes follow the way the material was presented in the lecture rather closely. Additions (e.g. from exercise sheets) and slight modifications have been marked with †.

Cut off for the exam is Christmas.

Contents

[Lecture 01, 2023-10-16]

Literature

- Schindler, Set theory
- K. Kunen
- T. Jech
- A. Kanamori, The higher infinite

Outline

- Set theory
  - Naive set theory
  - ZFC
  - Ordinals and Cardinals
  - Models of set theory (in particular forcing)
  - Independence of CH.

1 Naive set theory

Definition 1.1. Let $A \neq \emptyset$, B be arbitrary sets. We write $A \leqslant B$ (A is not bigger than B) iff there is an injection $f\colon A \hookrightarrow B$.

Lemma 1.2. If $A \leqslant B$, then there is a surjection $g\colon B \twoheadrightarrow A$.

Proof. Fix $f\colon A \hookrightarrow B$. If f is also surjective, then $f^{-1}\colon B \to A$ is also a bijection. Otherwise define g by choosing an arbitrary $x_0 \in B$ and let

$$g(y) := \begin{cases} x & : f(x) = y, \\ x_0 & : \text{if there is no such } x. \end{cases}.$$

□

Lemma 1.3. If there is a surjection $f\colon A \twoheadrightarrow B$, then $B \leqslant A$.

Proof. For every $x \in B$ choose one of its preimages under f. This is basically equivalent to AC. □

Definition 1.4. For sets A, B write $A < B$ iff $A \leqslant B \wedge B \nleqslant A$.

Theorem 1.5 (Cantor)**.** $\mathbb{N} < \mathbb{R}$.

Proof (Cantor's original proof). Clearly $\mathbb{N} \leqslant \mathbb{R}$. Take some function $f \colon \mathbb{N} \to \mathbb{R}$

Define a sequence $([a_n, b_n], n \in \mathbb{N})$ of nonempty closed nested intervals, i.e. $a_n \leqslant a_{n+1} < b_{n+1} \leqslant b_n$ as follows: Set $a_0 := 0$, $b_0 := 1$, and a_{n+1}, b_{n+1} such that $x_n \notin [a_{n+1}, b_{n+1}]$. Then $\bigcap_{n\in\mathbb{N}} [a_n, b_n] \neq \varnothing$ since $\mathbb{R}$ is complete. Thus f is not surjective. □

Notation 1.5.1. For a set A, $\mathbb{P}(A)$ denotes the **power set** of A, i.e. the set of all subsets of A.

Theorem 1.6. For all sets A, $A < \mathbb{P}(A)$.

Proof. Clearly $A \leqslant \mathbb{P}(A)$ since $A \ni a \mapsto \{a\} \in \mathbb{P}(A)$ is an injection.

Let $f \colon A \to \mathbb{P}(A)$, we want to show that this is not surjective. Let $c := \{x \in A | x \notin f(x)\} \in \mathbb{P}(A)$. Suppose that $f(x_0) = c$. Then both $x_0 \in c$ and $x_0 \notin c$ lead to a contradiction. □

Definition 1.7. For sets A, B write $A \sim B$ for $A \leqslant B$ and $B \leqslant A$.

Theorem 1.8 (Schröder-Bernstein)**.** Let A, B be any sets. If $A \sim B$, there is a bijection $h \colon A \to B$.

Proof. Let $f \colon A \hookrightarrow B$ and $g \colon B \hookrightarrow A$ be injective. We need to define a bijection $h \colon A \to B$. For each $x \in A$ we define $N(x) \in \mathbb{N} \cup \{\infty\}$ and the maximal "preimage sequence" $(x_n : n < N(x))$ as follows: $x_0 := x$, if $n + 1 < N$ and n is even, then $x_n := g(x_{n+1})$, if it is odd, $x_n := f(x_{n+1})$ and either $N = \infty$ or x_{N-1} has no preimage under f if $N - 1$ is even, resp. g if $N - 1$ is odd.

Similarly for each $y \in B$ an $M = M(y) \in \mathbb{N} \cup \{\infty\}$ and the maximal preimage sequence $(y_n : n < M)$ can be defined.

Let $A^{\text{odd}} := \{x \in A : N(x) \text{ is an odd natural number}\}$, $A^{\text{even}} := \{x \in A : N(x) \text{ is an even natural number}\}$, $A^{\infty} := \{x \in A : N(x) = \infty\}$ and similarly for B.

Now define

$$
\begin{aligned}
h\colon A &\longrightarrow B\\
x &\longmapsto \begin{cases} f(x) & : x \in A^{\text{odd}} \cup A^{\infty},\\ g^{-1}(x) & : x \in A^{\text{even}}.\end{cases}
\end{aligned}
$$

It is clear that this is bijective.

missing picture $f(A^{\text{odd}}) \subseteq B^{\text{even}}$, $f(A^{\infty}) = B^{\infty}$

. □

Definition 1.9. The **continuum hypothesis** (CH) says that there is no set A such that $\mathbb{N} < A < \mathbb{R}$, i.e. every uncountable subset $A \subseteq \mathbb{R}$ is in bijection with $\mathbb{R}$.

CH is equivalent to the statement that there is no set $A \subseteq \mathbb{R}$ which is uncountable ($\mathbb{N} < A$) and there is no bijection $A \leftrightarrow \mathbb{R}$.

What we'll do next: Define open and closed subsets of $\mathbb{R}$. Show CH for open and closed sets.

[Lecture 02, 2023-10-19]

Definition 1.10. A set $O \subseteq \mathbb{R}$ is called **open** in $\mathbb{R}$ iff it is the union of a set of open intervals.

A set $A \subseteq \mathbb{R}$ is called **closed** in $\mathbb{R}$ iff it is the complement of an open set.

Remark 1.10.2.

- If $\varnothing \neq O \overset{\text{open}}{\subseteq} \mathbb{R}$ then $O \sim \mathbb{R}$.
- If $O \subseteq \mathbb{R}$ is open, then O is the union of open intervals with rational endpoints, since $\mathbb{Q}$ is dense.

Remark† 1.10.3. $\{O \subseteq \mathbb{R}\} \sim 2^{\aleph_0} < \mathcal{P}(\mathbb{R})$.

Definition 1.11. We call $x \in \mathbb{R}$ an **accumulation point** of A iff for all $a < x < b$ there is some $y \in A$, $y \in (a,b)$, $y \neq x$. We write A' for the set of all accumulation points of A.

Example 1.12. $\{\frac{1}{n+1} | n \in \mathbb{N}\}' = \{0\}$.

Lemma 1.13. A set $A \subseteq \mathbb{R}$ is closed iff $A' \subseteq A$.

Proof of Lemma 1.13. "$\Longrightarrow$" Let A be closed. Suppose that $x \in A' \setminus A$. Then there exists $(a,b) \ni x$ disjoint from A. Hence $x \notin A'$ ↯

"$\Longleftarrow$" Suppose $A' \subseteq A$.

Claim 1.13.1. *$A \subseteq \mathbb{R}$ is closed iff all Cauchy sequences in A converge in A.*

Subproof. Let A be closed and $\langle x_n : n \in \omega \rangle$ a Cauchy sequence in A. Suppose that $x = \lim\limits_{n \to \infty} x_n \notin A$. Then there is $(a, b) \ni x$ disjoint from A. However $x_n \in (a, b)$ for almost all $n \in \omega$ $\lightning$

On the other hand let A not be closed. Then there exists a witness $x \in \mathbb{R} \setminus A$ such that $A \cap (a, b) \neq \varnothing$ for all $(a, b) \ni x$. In particular, we may pick $x_n \in (x - \frac{1}{n+1}, x + \frac{1}{n+1}) \cap A$ for all $n < \omega$. ■

Now if $A' \subseteq A$ and A were not closed, there would be some Cauchy-sequence (x_n) in A such that $\lim\limits_{n \to \infty} x_n \notin A$. But then $x \in A' \subseteq A \lightning$. □

Definition 1.14. $P \subseteq \mathbb{R}$ (or, more generally, a subset of any topological space) is called **perfect** iff $P \neq \varnothing$ and $P = P'$.

Example† 1.14.4. Note that being perfect depends on the surrounding topological space: For example, $[0, 1] \cap \mathbb{Q}$ is perfect as a subset of $\mathbb{Q}$, but not perfect as a subset of $\mathbb{R}$.

We want to prove two things:

- If P is perfect, then $P \sim \mathbb{R}$.
- If A is closed and uncountable then A has a perfect subset. In particular $A \sim \mathbb{R}$.

Lemma 1.15. Let $P \subseteq \mathbb{R}$ be perfect. Then $P \sim \mathbb{R}$.

Proof. It suffices to find an injection $f \colon \mathbb{R} \hookrightarrow P$. We have $\underbrace{\{0, 1\}^\omega}_{\text{infinite 0-1-sequences}} \sim \mathbb{R}$, hence it suffices to construct $f \colon \{0, 1\}^\omega \hookrightarrow P$.

In order to do that, we are going to construct some $g \colon \underbrace{\{0, 1\}^{<\omega}}_{\text{finite 0-1-sequences}} \to P$ with certain properties by recursion on the length of $s \in \{0, 1\}^{<\omega}$.

Let $g(\varnothing)$ be any point in P. Suppose that $g(s) \in P$ has been chosen for all s of length $\leqslant n$. For each $s \in \{0, 1\}^n$ pick $g(s) \in (a_s, b_s)$ such that $(a_s, b_s) \cap (a_{s'}, b_{s'}) = \varnothing$ for all s, s' of length n, $b_s - a_s \leqslant \frac{1}{n^3}$ and $(a_{s|_{n-1}}, b_{s|_{n-1}}) \subseteq (a_s, b_s)$.

For each such s pick $x_s \in (a_s, b_s) \cap P$ with $x_s \neq f(s)$. This is possible since $P \subseteq P'$. Now set $g(s^\frown 0) := g(s)$ and $g(s^\frown 1) := g(x_s)$. This finishes the construction.

If $t \in \{0,1\}^\omega$, then $(g(t|_n), n < \omega)$ is a Cauchy sequence.

By $P' \subseteq P$ we get that this sequence converges to a point in P. Define $f(t)$ to be this point.

If $t \neq t' \in \{0,1\}^\omega$, then there is some n such that $t|_n \neq t'|_n$, hence $f(t) \in [a_{t|_n}, b_{t|_n}]$ and $f(t') \in [a_{t'|_n}, b_{t'|_n}]$ which are disjoint. Thus $f(t) \neq f(t')$, i.e. f is injective. □

[Lecture 03, 2023-10-23]

Theorem 1.16 (Cantor-Bendixson)**.** If $A \subseteq \mathbb{R}$ is closed, it is either at most countable or else A contains a perfect set.

Corollary 1.17. If $A \subseteq \mathbb{R}$ is closed, then either $A \preccurlyeq \mathbb{N}$ or $A \sim \mathbb{R}$.

Fact 1.17.5. $A' = \{x \in \mathbb{R} | \forall a < x < b.\ (a,b) \cap A \text{ is at least countable}\}$.

Proof. $\supseteq$ is clear. For $\subseteq$, fix $a < x < b$ and let us define $(y_n : n \in \omega)$ as well as $((a_n, b_n) : n \in \omega)$. Set $a_0 := a$, $b_0 := b$. Having defined (a_n, b_n), pick $x \neq y_n \in A \cap (a_n, b_n)$, Then pick $a_n < a_{n+1} < x < b_{n+1} < b_n$ such that $y_n \notin (a_{n+1}, b_{n+1})$. Clearly $y_n \neq y_{n+1}$, hence $\{y_n : n \in \mathbb{N}\}$ is a countable subset of $A \cap (a,b)$. □

Definition 1.18. Let $A \subseteq \mathbb{R}$. We say that $x \in \mathbb{R}$ is a **condensation point** of A iff for all $a < x < b$, $(a,b) \cap A$ is uncountable.

By the fact we just proved, all condensation points are accumulation points.

Proof of Theorem 1.16. Fix $A \subseteq \mathbb{R}$ closed. We want to see that A is at most countable or there is some perfect $P \subseteq A$. Let

$$P := \{x \in \mathbb{R} | x \text{ is a condensation point of } A\}.$$

Since A is closed, $P \subseteq A$.

Claim 1.16.1. *$A \setminus P$ is at most countable.*

Subproof. For each $x \in A \setminus P$, there is $a_x < x < b_x$ such that $(a_x, b_x) \cap A$ is at most countable. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we may assume that $a_x, b_x \in \mathbb{Q}$.

Then

$$A \setminus P = \bigcup_{x \in A \setminus P} (a_x, b_x) \cap A.$$

$\subseteq$ holds by the choice of a_x and b_x. For $\supseteq$ let y be an element of the RHS. Then $y \in (a_{x_0}, b_{x_0}) \cap A$ for some x_0. As $(a_{x_0}, b_{x_0}) \cap A$ is at most countable, $y \notin P$.

Now we have that $A \setminus P$ is a union of at most countably many sets, each of which is at most countable. ■

Claim 1.16.2. *If $P \neq \varnothing$, the P is perfect.*

Subproof. $P \neq \varnothing$: ✓

$P \subseteq P'$:

Let $x \in P$. Let $a < x < b$. We need to show that there is some $y \in (a,b) \cap P \setminus \{x\}$. Suppose that for all $y \in (a,b) \setminus \{x\}$ there is some $a_y < y < b_y$ with $(a_y, b_y) \cap A$ being at most countable. Wlog. $a_y, b_y \in \mathbb{Q}$. Then

$$(a,b) \cap A = \{x\} \cup \bigcup_{\substack{y \in (a,b) \\ y \neq x}} [(a_y, b_y) \cap A].$$

But then $(a,b) \cap A$ is at most countable contradicting $x \in P$.

$P' \subseteq P$ (i.e. P is closed): Let $x \in P'$. Then for $a < x < b$ the set $(a,b) \cap P$ always has a member y such that $y \neq x$. Since $y \in P$, we get that $(a,b) \cap A$ in uncountable, hence $x \in P$. ■

But now

$$A = \overbrace{P}^{\text{perfect, unless } = \varnothing} \cup \underbrace{(A \setminus P)}_{\text{at most countable}}.$$

□

Alternative proof of Cantor-Bendixson

[Lecture 04,]

2 ZFC

ZFC stands for

- Zermelo's axioms (1905),
- Fraenkel's axioms,
- the Axiom of Choice (2.9).

Notation 2.0.6. We write $x \subseteq y$ as a shorthand for $\forall z.\ (z \in x \implies z \in y)$.

We write $x = \varnothing$ for $\neg \exists y. y \in x$ and $x \cap y = \varnothing$ for $\neg \exists z.\ (z \in x \wedge z \in y)$.

We use $x = \{y, z\}$ for

$$y \in x \wedge z \in x \wedge \forall a.\ (a \in x \implies a = y \vee a = z).$$

We write $z = x \cap y$ for

$$\forall u.\ ((u \in z) \implies u \in x \wedge u \in y),$$

$z = x \cup y$ for

$$\forall u.\ ((u \in z) \iff (u \in x \vee u \in y)),$$

$z = \bigcap x$ for

$$\forall u.\ ((u \in z) \iff (\forall v.\ (v \in x \implies u \in v))),$$

$z = \bigcup x$ for

$$\forall u.\ ((u \in z) \iff \exists v.\ (v \in x \wedge u \in v))$$

and $z = x \backslash y$ for

$$\forall u.\ ((u \in z) \iff (u \in x \wedge u \notin y)).$$

ZFC consists of the following axioms:

Axiom 2.1 (**Extensionality**)**.**

$$\forall x.\ \forall y.\ (x = y \iff \forall z.\ (z \in x \iff z \in y)).$$

Equivalent statements using $\subseteq$:

$$\forall x.\ \forall y.\ (x = y \iff (x \subseteq y \wedge y \subseteq x)).$$

Axiom 2.2 (**Foundation**)**.** Every set has an $\in$-minimal member:

$$\forall x.\ \left(\exists a.\ (a \in x) \implies \exists y.\ y \in x \wedge \neg \exists z.\ (z \in y \wedge z \in x)\right).$$

Shorter:

$$\forall x.\ (x \neq \varnothing \implies \exists y \in x.\ x \cap y = \varnothing).$$

Axiom 2.3 (**Pairing**)**.**

$$\forall x.\ \forall y.\ \exists z.\ (z = \{x, y\}).$$

Remark 2.3.7. Together with the axiom of pairing, the axiom of foundation implies that there can not be a set x such that $x \in x$: Suppose that $x \in x$. Then x is the only element of $\{x\}$, but $x \cap \{x\} \neq \varnothing$.

A similar argument shows that chains like $x_0 \in x_1 \in x_2 \in x_0$ are ruled out as well.

Axiom 2.4 (**Union**).
$$\forall x.\ \exists y.\ (y = \bigcup x).$$

Axiom 2.5 (**Power Set**). We write $x = \mathcal{P}(y)$ for $\forall z.\ (z \in x \iff z \subseteq x)$. The power set axiom states

$$\forall x.\ \exists y.\ y = \mathcal{P}(x).$$

Axiom 2.6 (**Infinity**). A set x is called **inductive**, iff $\varnothing \in x \wedge \forall y.\ (y \in x \implies y \cup \{y\} \in x)$.

The axiom of infinity says that there exists and inductive set.

Axiom Schema 2.7 (**Separation**). Let φ be some fixed fist order formula in $\mathcal{L}_\in$ with free variables $x, v_1, \dots, v_p$. Let b be a variable that is not free in φ. Then $(\mathrm{Aus})_\varphi$ states

$$\forall v_1.\ \forall v_p.\ \forall a.\ \exists b.\ \forall x.\ (x \in b \implies x \in a \wedge \varphi(x, v_1, v_p))$$

Let us write $b = \{x \in a | \varphi(x)\}$ for $\forall x.\ (x \in b \iff x \in a \wedge f(x))$. Then (Aus) can be formulated as

$$\forall a.\ \exists b.\ (b = \{x \in a | \varphi(x)\}).$$

Remark 2.7.8. (Aus) proves that

- $\forall a.\ \forall b.\ \exists c.\ (c = a \cap b)$,
- $\forall a.\ \forall b.\ \exists c.\ (c = a \backslash b)$,
- $\forall a.\ \exists b.\ (b = \bigcap a)$.

Axiom Schema 2.8 (**Replacement** (Fraenkel)). Let φ be some $\mathcal{L}_\in$ formula with free variables x, y. Then

$$\forall v_1 \dots \forall v_p.$$
$$[(\forall x \exists! y.\ \varphi(x, y, \overline{v})) \to \forall a.\ \exists b.\ \forall y.\ (y \in b \leftrightarrow \exists x (x \in a \wedge \varphi(x, y, \overline{v}))]$$

Axiom 2.9 (**Choice**). Every family of pairwise disjoint non-empty sets has a **choice set**:

$$\begin{aligned}&\forall x.\ (\\ &\quad ((\forall y \in x.\ y \neq \varnothing) \wedge (\forall y \in x.\ \forall y' \in x.\ (y \neq y' \implies y \cap y' = \varnothing)))\\ &\quad \implies \exists z.\ \forall y \in x.\ \exists u.\ (z \cap y = \{u\})\\ &\quad)\end{aligned}$$

[Lecture 05, 2023-10-30]

Definition 2.10. Zermelo:

$$\mathsf{Z} := \text{(Ext)} + \text{(Fund)} + \text{(Pair)} + \text{(Union)} + \text{(Pow)} + \text{(Inf)} + \text{(Aus)}_\varphi$$

Zermelo and Fraenkl:

$$\mathsf{ZF} := Z + \text{(Rep)}_\varphi$$

$$\mathsf{ZFC} := \mathsf{ZF} + \text{(C)}$$

Variants:

$$\mathsf{ZFC}^- := \mathsf{ZFC} \setminus \text{(Pow)}.$$

$$\mathsf{ZFC}^{-\infty} := \mathsf{ZFC} \setminus \text{(Inf)}$$

Definition 2.11. For sets x, y we write (x, y) for $\{\{x\}, \{x, y\}\}$.

Remark 2.11.9. Note that $(x, y) = (a, b) \iff x = a \wedge y = b$. ZFC proves that (x, y) always exists.

Definition 2.12. For sets $x_1, \ldots, x_{n+1}$ we write

$$(x_1, \ldots, x_{n+1}) := ((x_1, \ldots, x_n), x_{n+1})$$

where we assume that $(x_1, \ldots, x_n)$ is already defined.

Definition 2.13. The **cartesian product** $a \times b$ of two sets a and b is defined to be $a \times b := \{(x, y) | x \in a \wedge y \in b\}$.

Fact 2.13.10. $a \times b$ exists.

Proof. Use (Aus) over $\mathcal{P}(\mathcal{P}(a \cup b))$. □

Definition 2.14. For $a_1, \ldots, a_n$ we define

$$a_1 \times \ldots \times a_n := (a_1 \times \ldots \times a_{n-1}) \times a_n.$$

recursively.

For $a = a_1 = \ldots = a_n$, we write a^n for $a_1 \times \ldots \times a_n$.

Remark 2.14.11. The fact that ZFC can be used to encode all of mathematics, should not be overestimated. It is clumsy to do it that way. Nobody cares anymore. There are better foundations. What makes ZFC special is that it allows to investigate infinity.

Definition 2.15. An **n-ary relation** R is a subset of $a_1 \times \ldots \times a_n$ for some sets $a_1, \ldots, a_n$.

For a **binary relation** R (i.e. $n = 2$) we define

$$\operatorname{dom}(R) := \{x | \exists y.\ (x, y) \in R\}$$

and

$$\operatorname{ran}(R) := \{y | \exists x.\ (x, y) \in R\}.$$

Definition 2.16. A binary relation R is a **function** iff

$$\forall x \in \operatorname{dom}(R).\ \exists y.\ \forall y'.\ (y' = y \iff xRy').$$

A function f is a function from d to b iff $d = \operatorname{dom}(f)$ and $\operatorname{ran}(f) \subseteq b$.

We write $f \colon d \to b$. The set of all function from d to b is denoted by ${}^{d}b$ or b^d.

Fact 2.16.12. Given sets d, b then ${}^{d}b$ exists.

Proof. Apply again (Aus) over $\mathcal{P}(d \times b)$. ☐

Definition 2.17. We all know how **injective**, **surjective**, **bijective**, ... are defined.

Notation 2.17.13. For $f \colon d \to b$ and $a \subseteq d$ we write $f''a := \{f(x) : x \in a\}$ (the **pointwise image** of a under f).

(In other mathematical fields, this is sometimes denoted as $f(a)$. We don't do that here.)

Definition 2.18. A binary relation $\leqslant$ on a set a is a **partial order** iff $\leqslant$ is

- **reflexive**, i.e. $x \leqslant x$,
- **antisymmetric** (sometimes this is also called **symmetric**), i.e. $x \leqslant y \wedge x \leqslant y \implies x = y$, and
- **transitive**, i.e. $x \leqslant y \wedge y \leqslant z \implies x \leqslant z$.

If additionally $\forall x, y.\ (x \leqslant y \vee y \leqslant x)$, $\leqslant$ is called a **linear order** (or **total order**).

Definition 2.19. Let $(a, \leqslant)$ be a partial order. Let $b \subseteq a$. We say that x is a **maximal element** of b iff

$$x \in b \wedge \neg \exists y \in b.\ (y > x).$$

We say that x is the **maximum** of b, $x = \max(b)$, iff

$$x \in b \wedge \forall y \in b.\ y \leqslant x.$$

In a similar way we define **minimal elements** and the **minimum** of b. We say that x is an **upper bound** of b if $\forall y \in b.\ (x \geqslant y)$. Similarly **lower bounds** are defined.

We say $x = \sup(b)$ if x is the minimum of the set of upper bounds of b. (This does not necessarily exist.) Similarly $\inf(b)$ is defined.

Remark† 2.19.14. Note that in a partial order, a maximal element is not necessarily a maximum. However for linear orders these notions coincide.

Definition 2.20. Let $(a, \leqslant_a)$ and $(b, \leqslant_b)$ be two partial orders. Then a function $f\colon a \to b$ is called **order-preserving** iff

$$\forall x, y \in a.\ (x \leqslant_a y) \iff f(x) \leqslant_b f(y).$$

An order-preserving bijection is called an **isomorphism**. We write $(a, \leqslant_a) \cong (b, \leqslant_b)$ if they are isomorphic.

Definition 2.21. Let $(a, \leqslant)$ be a partial order. Then $(a, \leqslant)$ is a **well-order**, iff

$$\forall b \subseteq a.\ b \neq \varnothing \implies \min(b) \text{ exists.}$$

Fact 2.21.15. Let $(a, \leqslant)$ be a well-order, then $(a, \leqslant)$ is total.

Proof. For $x, y \in a$ consider $\{x, y\}$. Then $\min(\{x, y\}) \leqslant x, y$. □

Lemma 2.22. Let $(a, \leqslant)$ be a well-order. Let $f \colon a \to a$ be an order-preserving map. Then $f(x) \geqslant x$ for all $x \in a$.

Proof. Consider $x_0 := \min(\{x \in a | f(x) < x\})$. □

Lemma 2.23. If $(a, \leqslant)$ is a well-order and $f \colon (a, \leqslant) \leftrightarrow (a, \leqslant)$ is an isomorphism, then f is the identity.

Proof. By the last lemma, we know that $f(x) \geqslant x$ and $f^{-1}(x) \geqslant x$. □

Lemma 2.24. Suppose $(a, \leqslant_a)$ and $(b, \leqslant_b)$ are well-orderings such that $(a, \leqslant_a) \cong (b, \leqslant_b)$. Then there is a unique isomorphism $f \colon a \to b$.

Proof. Let f, g be isomorphisms and consider $g^{-1} \circ f \colon (a, \leqslant_a) \xrightarrow{\cong} (a, \leqslant_a)$. We have already shown that $g^{-1} \circ f$ must be the identity, so $g = f$. □

Definition 2.25. If $(a, \leqslant)$ is a partial order and if $x \in a$, then write $(a, \leqslant)|_x$ for $(\{y \in a | y \leqslant x\}, \leqslant \cap \{y \in a | y \leqslant x\}^2)$.

Abuse of Notation† **2.25.16.** For a partial order $(a, \leqslant_a)$ we sometimes just write a.

Theorem 2.26. Let $(a, \leqslant_a)$ and $(b, \leqslant_b)$ be well-orders. Then exactly one of the following three holds:

(i) $a \cong b$,

(ii) $\exists x \in b.\ a \cong b|_x$,

(iii) $\exists x \in a.\ a|_x \cong b$.

Proof. Let us define a relation $r \subseteq a \times b$ as follows: Let $(x, y) \in r$ iff $a|_x \cong b|_y$. By the previous lemma, for each $x \in a$, there is at most one $y \in b$ such that $(x, y) \in r$ and vice versa, so r is an injective function from a subset of a to a subset of b.

Claim 1. *r is order-preserving:*

Subproof. If $x <_a x'$, then consider the unique y' such that $a|_{x'} \cong b|_{y'}$. The isomorphism restricts to $a|_x \cong b|_y$ for some $y <_b y'$. ■

Claim 2. $\operatorname{dom}(r) = a \vee \operatorname{ran}(r) = b$.

Subproof. Suppose that $\operatorname{dom}(r) \subsetneq a$ and $\operatorname{ran}(r) \subsetneq b$.

Let $x := \min(a \setminus \operatorname{dom}(r))$ and $y := \min(b \setminus \operatorname{ran}(r))$. Then $(a, \leqslant_a)|_x \cong (b, \leqslant_b)|_y$. But now $(x, y) \in r$ which is a contradiction. ■

□

[Lecture 06, 2023-11-06]

Theorem 2.27 (Zorn)**.** Let $(a, \leqslant)$ be a partial order with $a \neq \varnothing$. Assume that for all $b \subseteq a$ with $b \neq \varnothing$ and b linearly ordered, b has an upper bound. Then a has a maximal element.

Proof of Theorem 2.27. Fix $(a, \leqslant)$ as in the hypothesis. Let $A := \{\{(b, x) : x \in b\} : b \subseteq a, b \neq \varnothing\}$. Note that A is a set (use separation on $\mathcal{P}(\mathcal{P}(a) \times \bigcup \mathcal{P}(a))$). Note further that if $b_1 \neq b_2$, then $\{(b_1, x) : x \in b_1\}$ and $\{(b_2, x) : x \in b_2\}$ are disjoint. Hence the Axiom of Choice (2.9) gives us a choice function f on A, i.e. $\forall b \in \mathcal{P}(a) \setminus \{\varnothing\}. \ (f(b) \in b)$.

Now define a binary relation $\leqslant^*$: We let W denote the set of all well-orderings $\leqslant'$ of subsets $b \subseteq a$, such that for all $u, v \in b$ if $u \leqslant' v$ then $u \leqslant v$ and for all $u \in b$ and

$$B_u^{\leqslant'} := \{w \in a : w \text{ is an } \leqslant\text{-upper bound of } \{v \in b : v \leqslant' u\}\}$$

then $B_u^{\leqslant'} \neq \varnothing$ and $f(B_u^{\leqslant'}) = u$.

Claim 2.27.1. *If $\leqslant', \leqslant'' \in W$, then $\leqslant' \subseteq \leqslant''$ or $\leqslant'' \subseteq \leqslant'$.*

Subproof. Let $\leqslant' \in W$ be a well-ordering of $b \subseteq a$ and let $\leqslant'' \in W$ be a well-ordering on $c \subseteq a$. We know that wlog. $(b, \leqslant') \cong (c, \leqslant'')$ or $\exists v \in c. \ (b, \leqslant') \cong (c, \leqslant'')|_v$. Let $g \colon b \to c$ or $g \colon b \to c|_v$ be a witness. We want to show that $g = \mathrm{id}$. Suppose that $g \neq \mathrm{id}$. Let $u_0 \in b$ be $\leqslant'$-minimal such that $g(u_0) \neq u_0$. Writing $\overline{g} := g|_{\{w \in b : w <' u_0\}}$, then $(b, \leqslant')|_{u_0} \cong (c, \leqslant'')|_{g(u_0)}$ and $\overline{g}$ is in fact the identity on $\{w \in b | w \leqslant' u_0\}$ but this means $\{w \in b | w <' u_0\} = \{w \in c | w <'' g(u_0)\}$ and $B_{u_0}^{\leqslant'} = B_{g(u_0)}^{\leqslant''} \neq \varnothing$. Then $u_0 = f(B_{u_0}^{\leqslant'}) = f(B_{g(u_0)}^{\leqslant''}) = g(u_0)$. Thus g is the identity. ■

Given the claim, we can now see that $\bigcup W$ is a well-order $\leqslant^{**}$ of a. Let $B = \{w \in a | w \text{ is a } \leqslant\text{-upper bound of } b\}$ (this is not empty by the hypothesis).

Suppose that b does not have a maximum. Then $B \cap b = \varnothing$. Now $f(B) = u_0$ and let

$$\leqslant^{**} = \leqslant^{*} \cup \{(u, u_0) | u \in b\} \cup \{(u_0, u_0)\}.$$

Then $B = B_{u_0}^{\leqslant^{**}}$. So $\leqslant^{**} \in W$, but now $u_0 \in b$. So b must have a maximum. Why does this prove the lemma?

$\square$

Remark 2.27.17. Over ZF the Axiom of Choice (2.9) and Zorn's Lemma (2.27) are equivalent.

Corollary 2.28 (Hausdorff's maximality principle)**.** Let $a \neq \varnothing$. Let $A \subseteq \mathcal{P}(a)$ be such that $\forall B \subseteq A$, if $x \subseteq y \vee y \subseteq x$ for all $x, y \in B$, then there is some $z \in A$ such that $x \subseteq z$ for all $x \in B$. Then A contains a $\subseteq$-maximal element.

Remark 2.28.18 (Cultural enrichment)**.** Other assertions which are equivalent to the Axiom of Choice (2.9):

- Every infinite family of non-empty sets $\langle a_i : i \in I\rangle$ has non-empty product, i.e.
$$\prod_{i \in I} a_i \neq \varnothing.$$
- Every set can be well-ordered.

2.1 The Ordinals

Goal. *We want to define nice representatives of the equivalence classes of well-orders.*

Recall that (Inf) states the existence of an inductive set x. We can hence form the smallest inductive set

$$\omega := \bigcap \{x : x \text{ is inductive}\}$$

Note that ω exists, as it is a subset of the inductive set given by (Inf). We call ω the set of **natural numbers**.

Notation 2.28.19. We write 0 for $\varnothing$, and $y+1$ for $y \cup \{y\}$.

With this notation the (Inf) is equivalent to

$$\exists x_0.\ (0 \in x_0 \wedge \forall n.\ (n \in x_0 \implies n+1 \in x_0)).$$

We have the following principle of induction:

Lemma 2.29. Let $A \subseteq \omega$ such that $0 \in A$ and for each $y \in A$, we have that $y+1 \in A$. Then $A = \omega$.

Proof. Clearly A is an inductive set, hence $\omega \subseteq A$. □

Definition 2.30. A set x is **transitive**, iff $\forall y \in x.\ y \subseteq x$.

Definition 2.31. A set x is called an **ordinal** (or **ordinal number**) iff x is transitive and for all $y, z \in x$, we have that $y = z$, $y \in z$ or $y \ni z$.

Clearly, the $\in$-relation is a well-order on an ordinal x.

Remark 2.31.20. This definition is due to John von Neumann.

Lemma 2.32. Each natural number (i.e. element of ω) is an ordinal.

Proof. We use Induction (2.29). Clearly $\varnothing$ is an ordinal. Now let α be an ordinal. We need to show that $\alpha + 1$ is an ordinal. It is transitive, since α is transitive and $\alpha \subseteq (\alpha + 1)$.

Let $x, y \in (\alpha + 1)$. If $x, y \in \alpha$, we know that $x = y \vee x \in y \vee x \ni y$ since α is an ordinal. Suppose $x = \alpha$. Then either $y = x$ or $y \in \alpha = x$. □

Lemma 2.33. ω is an ordinal.

Proof. ω is transitive:

Let $y \in \omega$. Let us show by Induction (2.29), that $y \subseteq \omega$. For $y = \emptyset$ this is clear.

Suppose that $y \in \omega$ with $y \subseteq \omega$. But now $\{y\} \subseteq \omega$, so $y + 1 = y \cup \{y\} \subseteq \omega$.

ω is well-ordered by $\in$:

We do a nested induction. First let

$$\varphi(y, z) := y \in z \lor y \ni z \lor y = z.$$

We want to show:

(a) $\varphi(0, 0)$

(b) $\forall z \in \omega.\ \varphi(0, z) \implies \varphi(0, z + 1)$.

(c) $\forall y \in \omega.\ ((\forall z' \in \omega.\ \varphi(y, z')) \implies (\forall z \in \omega.\ \varphi(y + 1, z)))$.

(a) and (b) are trivial. Fix $y \in \omega$ and suppose that $\forall z' \in \omega.\ \varphi(y, z')$. We want to show that $\forall z \in \omega.\ \varphi(y + 1, z)$.

We already know that $\forall z \in \omega.\ \varphi(0, z)$ holds by (b). In particular, $\varphi(0, y + 1)$ holds, so $\varphi(y + 1, 0)$ is true, since φ is symmetric. Now if $\varphi(y + 1, z)$ is true, we want to show $\varphi(y + 1, z + 1)$ is true as well. We have

$$(y + 1 \in z) \lor (y + 1 = z) \lor (y + 1 \ni z)$$

by assumption.

- If $y + 1 \in z \lor y + 1 = z$, then clearly $y + 1 \in z + 1$.
- If $y + 1 \ni z$, then either $z = y$ or $z \in y$.
  - In the first case, $z + 1 = y + 1$.
  - Suppose that $z \in y$. Then by the induction hypothesis $\varphi(y, z + 1)$ holds. If $y \in z + 1$, then $\{y, z\}$ would violate (Fund). If $y = z + 1$, then $z + 1 \in y + 1$. If $z + 1 \in y$, then $z + 1 \in y + 1$ as well.

□

[Lecture 07, 2023-11-09]

Notation 2.33.21. From now on, we will write $\alpha, \beta, \ldots$ for ordinals.

Lemma 2.34. (a) 0 is an ordinal, and if α is an ordinal, so is $\alpha + 1$.

(b) If α is an ordinal and $x \in \alpha$, then x is an ordinal.

(c) If α, β are ordinals and $\alpha \subseteq \beta$, then $\alpha = \beta$ or $\alpha \in \beta$.

(d) If α and β are ordinals, then $\alpha \in \beta$, $\alpha = \beta$ or $\alpha \ni \beta$.

Proof of Lemma 2.34. We have already proved (a) before.

(b) Fix $x \in \alpha$. Then $x \subseteq \alpha$. So if $y, z \in x$, then $y \in z \vee y = z \vee y \ni z$. Let $y \in x$. We need to see $y \subseteq x$. Let $z \in y$.

Claim 2.34.1. $z \in x$

Subproof. As α is transitive, we have that $z, y, x \in \alpha$. Thus $z \in x \vee z = x \vee z \ni x$.

$z = x$ contradicts (Fund): Consider $\{x, y\}$. Then $x \cap \{x, y\}$ is non empty, as it contains y. Furthermore $x \in y \cap \{x, y\}$

$z \ni x$ also contradicts (Fund): If $x \in z$, then $z \ni x \ni y \ni z \ni x \ni \ldots$. $\{x, y, z\}$ yields a contradiction, as $y \in x \cap \{x, y, z\}$, $z \in y \cap \{x, y, z\}$, $x \in z \cap \{x, y, z\}$.

So $z \in x$ as desired. ■

(c) Say $\alpha \subsetneq \beta$. Pick $\xi \in \beta \backslash \alpha$ such that $\eta \in \alpha$ for every $\eta \in \xi \cap \beta$. (This exists by (Fund)). We want to see that $\xi = \alpha$. We have $\xi \subseteq \alpha$ by the choice of ξ. On the other hand $\alpha \subseteq \xi$: Let $\eta \in \alpha \subseteq \beta$. We have that $\eta \in \xi \vee \eta = \xi \vee \eta \ni \xi$. If $\xi \in \eta$, then since $\eta \in \alpha$, we get $\xi \in \alpha$ contradicting the choice of ξ. If $\xi = \eta$, the $\xi = \eta \in \alpha$, which also is a contradiction. Thus $\eta \in \xi$.

This yields $\alpha \in \beta$, hence α is an ordinal.

(d) By (c) if α and β are ordinals, then $\alpha \subseteq \beta \iff (\alpha = \beta \vee \alpha \in \beta)$. We need tho see that if α, β are ordinals, then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. Suppose there are ordinals α, β such that this is not the case.

Pick such an α.

Let $\alpha_0 \in \alpha \cup \{\alpha\}$ be such that there is some β with $\neg(\beta \subseteq \alpha_0 \vee \alpha_0 \subseteq \beta)$ for for all $\gamma \in \alpha_0$, $\forall \beta.(\beta \subseteq \gamma \vee \gamma \subseteq \beta)$. Pick β_0 such that

$$\neg\left(\beta_0 \subseteq \alpha_0 \vee \alpha_0 \subseteq \beta_0\right).$$

Consider $\alpha_0 \cup \beta_0$.

Claim 2.34.2. *$\alpha_0 \cup \beta_0$ is an ordinal.*

Subproof. $\alpha_0 \cup \beta_0$ is clearly transitive. Let $\gamma, \delta \in \alpha_0 \cup \beta_0$. We claim that $\gamma \in \delta \vee \gamma = \delta \vee \gamma \in \delta$. This can only fail if $\gamma \in \alpha_0$ and $\delta \in \beta_0$ (or the other way around). But then $\gamma \in \delta \vee \gamma = \delta \vee \delta \in \gamma$ by the choice of α_0. ■

Claim 2.34.3. *$\alpha_0 = \alpha_0 \cup \beta_0$ or $\beta_0 = \alpha_0 \cup \beta_0$.*

Subproof. If that is not the case, then $\alpha_0 \in \alpha_0 \cup \beta_0$ and $\beta_0 \in \alpha_0 \cup \beta_0$. $\alpha_0 \in \alpha_0$ violates (Fund). Hence $\alpha_0 \in \beta_0$. By the same argument, $\beta_0 \in \alpha_0$. But this violates (Fund), as $\alpha_0 \in \beta_0 \in \alpha_0$. ■

□

Lemma 2.35. Let X be a set of ordinals, $X \neq \emptyset$. Then $\bigcap X$ and $\bigcup X$ are ordinals.

Proof. Easy. □

It is actually the case that $\bigcap X \in X$: Pick $\alpha \in X$ such that $\alpha \subseteq \beta$ for all $\beta \in X$. This exists by (Fund) and since all ordinals are comparable. Then $\alpha = \bigcap X$.

Notation 2.35.22. We write $\min(X)$ for $\bigcap X$ and $\sup(X)$ for $\bigcup X$.

It need not be the case that $\bigcup X \in X$, for example $\bigcup \omega = \omega$.

Definition 2.36. An ordinal α is called a **successor ordinal**, iff $\alpha = \beta \cup \{\beta\}$ for some $\beta \in \alpha$. Otherwise α is called a **limit ordinal**.

Observe. *Note that α is a limit ordinal iff for all $\beta \in \alpha$, $\beta + 1 \in \alpha$: If there is $\beta \in \alpha$ such that $\beta + 1 \notin \alpha$, then either $\alpha = \beta + 1$ (i.e. α is a successor) or $\alpha \in \beta + 1$, in which case $\beta \in \alpha \in \beta \cup \{\beta\}$ ↯.*

Also if α is a successor, then by definition there is some $\beta \in \alpha$, with $\beta + 1 = \alpha$, so $\beta + 1 \notin \alpha$.

Notation 2.36.23. If α, β are ordinals, we write $\alpha < \beta$ for $\alpha \in \beta$ (equivalently $\alpha \subsetneq \beta$). We also write $\alpha \leqslant \beta$ for $\alpha \in \beta \vee \alpha = \beta$ (i.e. $\alpha \subseteq \beta$).

Example 2.37. Limit ordinals:

- 0,
- ω,
- $\omega + \omega = \sup(\omega \cup \{\omega, \omega + 1, \ldots\})$,[a] $\omega + \omega + \omega, \omega + \omega + \omega + \omega, \ldots$

Successor ordinals:

- $1 = \{0\}, 2 = \{0, 1\}, 3, \ldots$
- $\omega + 1 = \omega \cup \{\omega\}, \omega + 2, \ldots,$

[a] To show that this exists, we need the recursion theorem and replacement.

[Lecture 08, 2023-11-13]

2.2 Classes

It is often very handy to work in a class theory rather than in set theory.

To formulate a class theory, we start out with a first order language with two types of variables, sets (denoted by lower case letters) and classes (denoted by capital letters), as well as one binary relation symbol $\in$ for membership.

Bernays-Gödel class theory (**BG**) has the following axioms:

Axiom 2.38 (Extensionality)**.**

$$\forall x.\ \forall y.\ \left(x = y \iff (\forall z.\ (z \in x \iff z \in y))\right).$$

Axiom 2.39 (Foundation)**.**

$$\forall x.\ (x \neq \varnothing \implies \exists y \in x.\ y \cap x = \varnothing).$$

Axiom 2.40 (Pairing)**.**

$$\forall x.\ \forall y.\ \exists z.\ z = \{x, y\}.$$

Axiom 2.41 (Union)**.**

$$\forall x.\ \exists y.\ y = \bigcup x.$$

Axiom 2.42 (Power Set)**.**

$$\forall x.\ \exists y.\ y = \mathcal{P}(x).$$

Axiom 2.43 (Infinity)**.**

$$\exists x.\ (\varnothing \in x \wedge (\forall y \in x.\ y \cup \{y\} \in x)).$$

Together with the following axioms for classes:

Axiom 2.44 (Extensionality for classes)**.**

$$\forall X.\ \forall Y.\ \left(\forall x.\ (x \in X \iff x \in Y) \implies X = Y\right).$$

Axiom 2.45. Every set is a class:

$$\forall x.\ \exists X.\ x = X.$$

Axiom 2.46. Every element of a class is a set:

$$\forall X.\ \exists Y.\ (X \in Y \to \exists x.\ x = X).$$

Axiom 2.47 (Replacement)**.** If F is a function and a is a set, then $F''a$ is a set.

Here a **(class) function** is a class consisting of pairs (x, y), such that for every x there is at most one y with $(x, y) \in F$. Furthermore $F''a := \{y : \exists x \in a.\ (x, y) \in F\}$.

Remark 2.47.24. Note that we didn't need to use an axiom schema, (Rep) is a single axiom.

Axiom 2.48 (Comprehension)**.**

$$\forall X_1.\ \ldots \forall X_k.\ \exists Y.\ (\forall x.\ x \in Y \iff \varphi(x, X_1, \ldots, X_k))$$

where $\varphi(x, X_1, \ldots, X_k)$ is a formula which contains exactly $X_1, \ldots, X_k, x$ as free variables, and φ does not have quantifiers ranging over classes.[a]

[a] If one removes the restriction regarding quantifiers, another theory, called **Morse-Kelly** set theory, is obtained.

notation: $\varnothing, \cap$

(The following was actually done in lecture 9, but has been moved here for clarity.)

BGC (in German often NBG[1]) is defined to be BG together with the additional axiom:

Axiom 2.49 (Choice)**.**

$$\exists F.\ (F \text{ is a function} \land \forall x \neq \varnothing. F(x) \in x).$$

Fact 2.49.25. BGC is conservative over ZFC, i.e. for all formulae φ in the language of set theory (only set variables) we have that if $\mathsf{BGC} \vdash \varphi$ then $\mathsf{ZFC} \vdash \varphi$.

We cannot prove this fact at this point, as the proof requires forcing. The converse is easy however, i.e. if $\mathsf{ZFC} \vdash \varphi$ then $\mathsf{BGC} \vdash \varphi$.

Notation 2.49.26. From now on, objects denoted by capital letters are (potentially proper) classes.

[1] **Neumann-Bernays-Gödel**

2.3 Induction and Recursion

Definition 2.50. A binary relation R on a set X, i.e. $R \subseteq X \times X$, is called **well-founded** iff for all $\emptyset \neq Y \subseteq X$ there is some $x \in Y$ such that for no $y \in Y.(y, x) \in R$.

Example 2.51. (a) $(\mathbb{N}, <)$ is well-founded.

(b) Let M be a set, and let $\in|_M := \{(x, y) : x, y \in M \wedge x \in y\}$. (Fund) is equivalent to saying that this is a well-founded relation for every M.

Lemma 2.52. In ZFC $-$(Fund), the following are equivalent:

- (Fund),
- There is no sequence $\langle x_n : n < \omega \rangle$ such that $x_{n+1} \in x_n$ for all $n < \omega$.

Proof. Suppose such sequence exists. Then $\{x_n : n < \omega\}$[2] violates (Fund).

For the other direction let $M \neq \emptyset$ be some set. Suppose that (Fund) does not hold for M.

Using (C), we construct an infinite sequence $x_0 \ni x_1 \ni x_2 \ni \ldots$ of elements of M.

More formally, for each $x \in M$ let $A_x := \{y \in M : y \in x\}$. Suppose that $A_x \neq \emptyset$ for all $x \in M$. Using (C) we get a function for $\langle A_x : x \in M \rangle$,[3] i.e. a function $f : M \to M$ such that $f(x) \in A_x$ for $x \in M$. Now fix $x \in M$. We want to produce a function $g : \omega \to M$ such that

- $g(0) = x$,
- $g(n+1) = f(g(n)) \in A_{g(n)}$.

Let

$$\begin{aligned} G = \{\overline{g} : \exists n \in \omega.& \\ &\overline{g} \text{ is a function with domain } n \text{ and range } \subseteq M, \text{ such that} \\ &\overline{g}(0) = x \wedge \forall m \in \omega.\ (m+1 \in \operatorname{dom}(\overline{g}) \implies \overline{g}(m+1) = f(\overline{g}(m)))\}. \end{aligned}$$

G exists as it can be obtained by (Aus) from ${}^{<\omega}M$. By induction, for every $n \in \omega$, there is a $\overline{g} \in G$ with $\operatorname{dom}(\overline{g}) \in n+1$: This holds for $n = 0$, as $\{(0, x)\} \in G$. If $\overline{g} \in G$ with $\operatorname{dom}(\overline{g}) = n+1$, then $\overline{g} \cup \{(n+1, f(\overline{g}(n)))\} \in G$. Also by induction, for every $n \in \omega$, there is a *unique* $\overline{g}$ with $\operatorname{dom}(\overline{g}) = n+1$.

Now let $g = \bigcup \overline{G}$. Also let $g(0) = x$ and $g(n+1) = f(g(n))$ for all $n \in \omega$. $\square$

[2]This exists as by definition the sequence (x_n) is a function $f : \omega \to V$ and this set is the image of f.

[3]Actually we only need the axiom of dependent choice, a weaker form of the Axiom of Choice (2.9). We'll discuss this later.

Lemma 2.53 (Dependent Choice)**.** Suppose that $M \neq \emptyset$ and R is a binary relation on M such that for all $x \in M$, $A_x := \{y \in M : (y, x) \in R\}$ is not empty.

Then for every $x \in M$ there exists a function $g \colon \omega \to M$ such that $g(0) = x$ and $g(n+1) \in A_{g(n)}$ for all $n < \omega$.

Proof. We showed a special case of this in the proof of Lemma 2.52. □

Remark 2.53.27. In ZF this is a weaker form of (C).

The construction of g in the previous proof was a special case of a construction on the proof of the recursion theorem:

[Lecture 09, 2023-11-16]

Definition 2.54. Let R be a binary relation. R is called **well-founded** iff for all classes X, there is an R-least y such that there is no $z \in X$ with $(z, y) \in R$.

Theorem 2.55 (Induction (again, but now for classes))**.** Suppose that R is a well-founded relation. Let X be a class such that for all sets x,

$$\{y : (y, x) \in R\} \subseteq X \implies x \in X.$$

Then X contains all sets.

Proof. Assume otherwise. Consider $Y = \{x : x \notin X\} \neq \emptyset$. By hypothesis, there is some $x \in Y$ such that $(y, x) \notin R$ for all $y \in Y$. In other words, if $(y, x) \in R$, then $x \notin Y$, i.e. $x \in X$. Thus $\{y : (y, x) \in R\} \subseteq X$. Hence $x \in X$ ↯. □

An alternative way of formulating this is

Theorem 2.56. Suppose R is a well-founded binary relation on A, i.e. $R \subseteq A \times A$. Suppose for all $\overline{A} \subseteq A$ is such that for all $x \in X$,

$$\{y \in A : (y, x) \in R\} \subseteq \overline{A} \implies x \in \overline{A}.$$

Then $\overline{A} = A$.

Definition 2.57. Let R be a binary relation. R is called **set-like** iff for all x, $\{y : (y, x) \in R\}$ is a set.

Theorem 2.58. Let R be a well-founded and set-like relation on A (i.e. $R \subseteq A \times A$).

Let D be a class of triples such that for all u, x there is exactly one y with $(u, x, y) \in D$ (basically $(u, x) \mapsto y$ is a function).

Then there is a unique function f on A such that for all $x \in A$,

$$(F|_{\{y \in A : (y,x) \in R\}}, x, F(x)) \in D,$$

i.e. $F(x)$ is computed from $F|_{\{y \in A : (y,x) \in R\}}$.

Proof. Uniqueness:

Let F, F' be two such functions. Suppose that $\overline{A} = \{x \in A : F(x) \neq F'(x)\} \neq \emptyset$. As R is well-founded, there is some $x \in \overline{A}$ such that $y \notin \overline{A}$ for all $y \in A$, $(y, x) \in R$. I.e. $F(y) = F'(y)$ for all $y \in A$, $(y, x) \in R$.

But then $F(x)$ is the unique y with $(F|_{\{z :\ (z,x) \in R\}}, x, y) \in D$, in particular it is the same as $F'(x)$ ↯

Existence:

Let us call a (set) function f *good*, if

- $\mathrm{dom}(f) \subseteq A$,
- if $x \in \mathrm{dom}(f)$ and $y \in A, (y, x) \in R$, then $y \in \mathrm{dom}(f)$ and
- for all $x \in \mathrm{dom}(f)$:

$$(f|_{\{y \in A : (y,x) \in R\}}, x, f(x)) \in D.$$

By the proof of uniqueness, we have that all good functions are coherent, i.e. $f(x) = f'(x)$ for good functions f, f' and all $x \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$. We may now let $F = \bigcup\{f : f \text{ is good}\}$, this exists by comprehension.

If $x \in \mathrm{dom}(F)$ and $y \in A$ with $(y, x) \in R$, then $y \in \mathrm{dom}(F)$ and

$$(F|_{\{y : (y,x) \in R\}}, x, F(x)) \in D.$$

We need to show that $\mathrm{dom}(F) = A$. This holds by induction: Suppose for a contradiction that $A \setminus \mathrm{dom}(F) \neq \emptyset$. Then there exists an R-least element x in this set, i.e.$x \notin \mathrm{dom}(F)$, but $y \in \mathrm{dom}(F)$ for all $(y, x) \in R$. For each $y \in A$ with $(y, x) \in R$, pick some good function f_y with $y \in \mathrm{dom}(f_y)$ Since R is set-like, we have that $f = \bigcup_y f_y$ is a good function. But then $f \cup (x, z)$, where z is unique such that $(f|_{\{y : (y,x) \in R\}}, x, z) \in D$, is good ↯. □

[Lecture 10,]

2.3.1 Applications of induction and recursion

Fact 2.58.28. For every set x there is a transitive set t such that $x \in t$.

Proof. Take $R = \in$. We want a function F with domain ω such that $F(0) = \{x\}$ and $F(n+1) = \bigcup F(n)$. Once we have such a function, $\{x\} \cup \bigcup \operatorname{ran}(F)$ is a set as desired. To get this F using the Recursion Theorem (2.58), pick D such that

$$(\varnothing, 0, \{x\}) \in D$$

and

$$(f, n+1, \bigcup\bigcup \operatorname{ran}(f)) \in D.$$

The Recursion Theorem (2.58) then gives a function such that

$$\begin{aligned} F(0) &= \{x\}, \\ F(n+1) &= \bigcup\bigcup \operatorname{ran}(F|_{n+1}) \\ &= \bigcup\bigcup \underbrace{\{\{x\}, x, \bigcup x, \dots, \bigcup^{n-1} x\}}_{F(n)} = \bigcup F(n), \end{aligned}$$

i.e. $F(n+1) = \bigcup F(n)$. □

Notation 2.58.29. Let OR denote the class of all ordinals and V the class of all sets.

Lemma 2.59. There is a function $F\colon \mathrm{OR} \to V$ such that $F(\alpha) = \bigcup\{\mathcal{P}(F(\beta)) : \beta < \alpha\}$.

Proof. Use the Recursion Theorem (2.58) with $R = \in$ and $(w, x, y) \in D$ iff

$$y = \bigcup\{\mathcal{P}(\overline{y}) : \overline{y} \in \operatorname{ran}(w)\}.$$

This function has the following properties:

$$\begin{aligned} &F(0) = \bigcup \varnothing = \varnothing, \\ &F(1) = \bigcup\{\mathcal{P}(\varnothing)\} = \bigcup\{\{\varnothing\}\} = \{\varnothing\}, \\ &F(2) = \bigcup\{\mathcal{P}(\varnothing), \mathcal{P}(\{\varnothing\})\} = \bigcup\{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{\varnothing, \{\varnothing\}\}, \\ &\quad \dots \end{aligned}$$

It is easy to prove by induction:

(a) Every $F(\alpha)$ is transitive.

(b) $F(\alpha) \subseteq F(\beta)$ for all $\alpha \leqslant \beta$.

(c) $F(\alpha+1) = \mathcal{P}(F(\alpha))$ for all $\alpha \in \mathrm{OR}$.

(d) $F(\lambda) = \bigcup\{F(\beta) : \beta < \lambda\}$ for $\lambda \in \mathrm{OR}$ a limit.

$\square$

Notation 2.59.30. Usually, one writes V_α for $F(\alpha)$. They are called the **rank initial segments** of V.

Lemma 2.60. If x is any set, then there is some $\alpha \in \mathrm{OR}$ such that $x \in V_\alpha$, i.e. $V = \bigcup\{V_\alpha : \alpha \in \mathrm{OR}\}$.

Proof. We use induction on the well-founded $\in$-relation. Let $A = \bigcup\{V_\alpha : \alpha \in \mathrm{OR}\}$. We need to show that $A = V$. By induction it suffices to prove that for every $x \in V$, if $\{y : y \in x\} \subseteq A$, then $x \in A$. The hypothesis says that for all $y \in x$, there is some α with $y \in V_\alpha$. Write α_y for the least such α. By (Rep), $\{\alpha_y : y \in x\}$ is a set and we may let $\alpha = \sup\{\alpha_y : y \in x\} \geqslant \alpha_y$ for all $y \in x$. Then $y \in V_{\alpha_y} \subseteq V_\alpha$ for all $y \in x$.

In other words $x \subseteq V_\alpha$, hence $x \in V_{\alpha+1}$. $\square$

Lemma 2.61 (Transitive collapse/Mostowski collapse). Let R be a binary set-like relation on a class A. Then R is well-founded iff there is a transitive class B such that

$$(B, \in|_B) \cong (A, R),$$

i.e. there is an isomorphism F, that is a function $F\colon B \to A$ with $x \in y \iff (F(x), F(y)) \in R$ for $x, y \in B$.

Proof. "$\Longleftarrow$" Suppose that R is ill-founded (i.e. not well-founded). Then there is some $(y_n : n < \omega)$ such that $y_n \in A$ and $(y_{n+1}, y_n) \in R$ for all $n < \omega$. But then if F is an isomorphism as above,

$$F^{-1}(Y_{n+1}) \in F^{-1}(Y_n)$$

for all $n < \omega$ $\lightning$

"$\Longrightarrow$" Suppose that R is well-founded. We want a transitive class B and a function $F\colon B \leftrightarrow A$ such that

$$x \in y \iff (F(x), F(y)) \in R.$$

Equivalently $G\colon A \leftrightarrow B$ with $(x,y) \in R$ iff $G(x) \in G(y)$ for all $x, y \in A$.

In other words, $G(y) = \{G(x) : (x,y) \in R\}$. Such a function G and class B exist by the Recursion Theorem (2.58). $\square$

As a consequence of the Mostowski Collapse (2.61), we get that if $<$ is a well-order on a set a then there is some transitive set b with $(b, \in|_b) \cong (a, <)$.

Lemma 2.62 (**Rank function**)**.** Let R be a well-founded and set-like binary relation on a class A. Then there is a function $F\colon A \to \mathrm{OR}$, such that for all $x, y \in A$

$$(x, y) \in R \implies F(x) < F(y).$$

Proof. By the Recursion Theorem (2.58), there is F such that

$$F(y) = \sup\{F(x) + 1 : (x, y) \in R\}.$$

This function is as desired. □

This does not skip any ordinals, as $F(y)$ is the least ordinal $> F(x)$ for all $(x, y) \in R$. Thus $\mathrm{ran}(F)$ is transitive. So either $\mathrm{ran}(F) = \mathrm{OR}$ or $\mathrm{ran}(F) \in \mathrm{OR}$. This F is called the **rank function** for (A, R).

Notation 2.62.31.

$$\mathrm{rk}_R(x) = \|x\|_R := F(x),$$

and

$$\mathrm{rank}(R) := \mathrm{ran}(F).$$

In the special case that R is a linear order on A, hence a well-order, $\mathrm{rank}(R)$ is called the **order type** of R (or of (A, R)), written $\mathrm{otp}(R)$.

[Lecture 11, 2023-11-23]

2.4 Cardinals

Definition 2.63. Let a be any set. The **cardinality** of a denoted by $\overline{\overline{a}}$, $|a|$ or $\mathrm{card}(a)$, is the smallest ordinal α such that there is some bijection $f\colon \alpha \to a$.

An ordinal α is called a **cardinal**, iff there is some set a with $|a| = \alpha$ (equivalently, $|\alpha| = \alpha$).

We often write $\kappa, \lambda, \ldots$ for cardinals.

Lemma 2.64. For every cardinal κ, there is come cardinal $\lambda > \kappa$.

Proof. Consider the powerset of κ. We know that there is no surjection $\kappa \twoheadrightarrow \mathcal{P}(\kappa)$. Hence $\kappa < |2^\kappa|$. □

Definition 2.65. For each cardinal κ, κ^+ denotes the least cardinal $\lambda > \kappa$.

Warning 2.66. This has nothing to do with the ordinal successor of κ.

Lemma 2.67. Let X be any set of cardinals. Then $\sup X$ is a cardinal.

Proof. If there is some $\kappa \in X$ with $\lambda \leqslant \kappa$ for all $\lambda \in X$, then $\kappa = \sup(X)$ is a cardinal.

Let us now assume that for all $\kappa \in X$ there is some $\lambda \in X$ with $\lambda > \kappa$. Suppose that $\sup(X)$ is not a cardinal and write $\mu = |\sup(X)|$. Then $\mu \in \sup(X)$, since $\sup(X)$ is an ordinal. However $\sup(X)$ is the least ordinal larger than all $\alpha \in X$, so there is $\lambda \in X$ with $\lambda > \mu$. However, there exists $\mu \twoheadrightarrow \sup(X)$, hence also $\mu \twoheadrightarrow \lambda$ (which is in contradiction to λ being a cardinal). □

We may now use the Recursion Theorem (2.58) to define a sequence $\langle \aleph_\alpha : \alpha \in \mathrm{OR} \rangle$ with the following properties:

$$\begin{aligned} \aleph_0 &= \omega, \\ \aleph_{\alpha+1} &= (\aleph_\alpha)^+, \\ \aleph_\lambda &= \sup\{\aleph_\alpha : \alpha < \lambda\}. \end{aligned}$$

Each $\aleph_\alpha$ is a cardinal. Also, a trivial induction shows that $\alpha \leqslant \aleph_\alpha$. In particular $|\alpha| \leqslant \aleph_\alpha$. Therefore the $\aleph_\alpha$ are all the infinite cardinals: If a is any infinite set, then $|a| \leqslant \aleph_{|a|}$, so $|a| = \aleph_\beta$ for some $\beta \leqslant |\alpha|$.

Notation 2.67.32. Sometimes we write ω_α for $\aleph_\alpha$ (when viewing it as an ordinal).

Notation 2.67.33. Let ${}^a b := \{f : f \text{ is a function}, \mathrm{dom}(f) = a, \mathrm{ran}(f) \subseteq b\}$.

Definition 2.68 (Cardinal arithmetic)**.** Let κ, λ be cardinals. Define

$$\begin{aligned} \kappa + \lambda &:= |\{0\} \times \kappa \cup \{1\} \times \lambda|, \\ \kappa \cdot \lambda &:= |\kappa \times \lambda|, \\ \kappa^\lambda &:= |{}^\lambda \kappa|. \end{aligned}$$

Warning 2.69. This is very different from ordinal arithmetic!

Theorem 2.70 (Hessenberg)**.** For all α we have

$$\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha.$$

Corollary 2.71. For all α, β it is

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}.$$

Proof. Wlog. $\alpha \leqslant \beta$. Trivially $\aleph_\alpha \leqslant \aleph_\beta$. It is also clear that

$$\aleph_\beta \leqslant \aleph_\alpha + \aleph_\beta \leqslant \aleph_\alpha \cdot \aleph_\beta \leqslant \aleph_\beta \cdot \aleph_\beta = \aleph_\beta.$$

□

Proof of Theorem 2.70. Define a well-order $<^*$ on $\mathrm{OR} \times \mathrm{OR}$ by setting

$$(\alpha, \beta) <^* (\gamma, \delta)$$

iff

- $\max(\alpha, \beta) < \max(\gamma, \delta)$ or
- $\max(\alpha, \beta) = \max(\gamma, \delta)$ and $\alpha < \gamma$ or
- $\max(\alpha, \beta) = \max(\gamma, \delta)$ and $\alpha = \gamma$ and $\beta < \delta$.

It is clear that this is a well-order.

There is an isomorphism

$$(\mathrm{OR}, <) \cong^{\Gamma^{-1}} (\mathrm{OR} \times \mathrm{OR}, <^*).$$

Γ is called the **Gödel pairing function**.

Claim 2.70.1. *For all α it is $\mathrm{ran}(\Gamma|_{\aleph_\alpha \times \aleph_\alpha}) = \aleph_\alpha$, i.e.*

$$\aleph_\alpha = \{\xi : \exists \eta, \eta' < \aleph_\alpha.\ \xi = \Gamma((\eta, \eta'))\}.$$

Subproof. We use induction of α. The claim is trivial for $\alpha = 0$. Now let $\alpha > 0$ and suppose the claim to be true for all $\beta < \alpha$. It is easy to see that

$$\mathrm{ran}(\Gamma|_{\aleph_\alpha \times \aleph_\alpha}) \supseteq \aleph_\alpha,$$

as otherwise $\Gamma|_{\aleph_\alpha \times \aleph_\alpha} : \aleph_\alpha \times \aleph_\alpha \to \eta$ would be a bijection for some $\eta < \aleph_\alpha$, but $\aleph_\alpha$ is a cardinal.

Suppose that $\mathrm{ran}(\Gamma|_{\aleph_\alpha \times \aleph_\alpha}) \supsetneq \aleph_\alpha$. Then there exist $\eta, \eta' < \aleph_\alpha$ with

$$\Gamma((\eta, \eta')) = \aleph_\alpha.$$

So $\Gamma|_{\{(\gamma,\delta):(\gamma,\delta)<^*(\eta,\eta')\}}$ is bijective onto $\aleph_\alpha$. If $(\gamma,\delta) <^* (\eta,\eta')$, then $\max\{\gamma,\delta\} \leqslant \max\{\eta,\eta'\}$. Say $\eta \leqslant \eta' < \aleph_\alpha$ and let $\aleph_\beta = |\eta'|$. There is a surjection

$$f\colon \underbrace{(\eta+1)}_{\leqslant\aleph_\beta} \times \underbrace{(\eta'+1)}_{\sim\aleph_\beta} \twoheadrightarrow \aleph_\alpha.$$

This gives rise to a surjection $f^*\colon \aleph_\beta \times \aleph_\beta \to \aleph_\alpha$. The inductive hypothesis then produces a surjection $f^*\colon \aleph_\beta \to \aleph_\alpha$ $\lightning$. ■

□

However, exponentiation of cardinals is far from trivial:

Observe. $2^\kappa = |\mathcal{P}(\kappa)|$, *since* ${}^\kappa\{0,1\} \leftrightarrow \mathcal{P}(\kappa)$.

Hence by Cantor $2^\kappa \geqslant \kappa^+$.

This is basically all we can say.

The **continuum hypothesis** states that $2^{\aleph_0} = \aleph_1$.

[Lecture 12, 2023-11-27]

2.5 Ordinal arithmetic

We define $+$, $\cdot$ and exponentiation for ordinals as follows:

Fix an ordinal β. We recursively define

$$\begin{aligned}
\beta + 0 &:= \beta \\
\beta + (\alpha+1) &:= (\beta+\alpha)+1, \\
\beta + \lambda &:= \sup_{\alpha<\lambda} \beta + \alpha \qquad \text{for limit ordinals } \lambda
\end{aligned}$$

(Recall that $\alpha + 1 = \alpha \cup \{\alpha\}$ was already defined.)

$$\begin{aligned}
\beta \cdot 0 &:= 0, \\
\beta \cdot (\alpha+1) &:= \beta\cdot\alpha + \beta, \\
\beta \cdot \lambda &:= \sup_{\alpha<\lambda} \beta\cdot\alpha \quad \text{for limit ordinals } \lambda
\end{aligned}$$

and

$$\begin{aligned}
\beta^0 &:= 1, \\
\beta^{\alpha+1} &:= \beta^\alpha \cdot \beta, \\
\beta^\lambda &:= \sup_{\alpha<\lambda} \beta^\alpha \quad \text{for limit ordinals } \lambda.
\end{aligned}$$

Example 2.72.

- $2 + 2 = 4$,
- $196883 + 1 = 196884$,
- $1 + \omega = \sup_{n<\omega} 1 + n = \omega \neq \omega + 1$,
- $2 \cdot \omega = \sup_{n<\omega} 2 \cdot n = \omega$,
- $\omega \cdot 2 = \omega \cdot 1 + \omega = \omega + \omega$.

Warning 2.73. Cardinal arithmetic and ordinal arithmetic are very different! The symbols are the same, but usually we will distinguish between the two by the symbols used for variables (i.e. $\alpha, \beta, \omega, \omega_1$ are viewed primarily as ordinals and $\kappa, \lambda, \aleph_\alpha$ as cardinals).

We will very rarely use ordinal arithmetic.

2.6 Cofinality

Definition 2.74. Let α, β be ordinals. We say that $f\colon \alpha \to \beta$ is **cofinal** iff for all $\xi < \beta$, there is some $\eta < \alpha$ such that $f(\eta) \geqslant \xi$.

Remark 2.74.34. If β is a limit ordinal, this is equivalent to

$$\forall \xi < \beta.\ \exists \eta < \alpha.\ f(\eta) > \xi.$$

Example 2.75. (a) Look at $\omega + \omega$.

$$\begin{aligned} f\colon \omega &\longrightarrow \omega + \omega \\ n &\longmapsto \omega + n \end{aligned}$$

is cofinal.

(b) Look at $\aleph_\omega$. Then

$$\begin{aligned} f\colon \omega &\longrightarrow \aleph_\omega \\ n &\longmapsto \aleph_n \end{aligned}$$

is cofinal.

Definition 2.76. Let β be an ordinal. The **cofinality** of β, denoted $\mathrm{cf}(\beta)$, is the least ordinal α such that there exists a cofinal $f\colon \alpha \to \beta$.

Example 2.77.
- $\mathrm{cf}(\aleph_\omega) = \omega$. In fact $\mathrm{cf}(\aleph_\lambda) \leqslant \lambda$ for limit ordinals $\lambda \neq 0$ (consider $\alpha \mapsto \aleph_\alpha$).
- $\mathrm{cf}(\aleph_{\omega+\omega}) = \omega$.

Lemma 2.78. For any ordinal β, $\mathrm{cf}(\beta)$ is a cardinal.

Proof. Let $f\colon \alpha \to \beta$ be cofinal. Then $\tilde{f}\colon |\alpha| \to \beta$, the composition with $\alpha \leftrightarrow |\alpha|$ is cofinal as well and $|\alpha| \leqslant \alpha$. ☐

Question 2.78.35. How does one imagine ordinals with cofinality $> \omega$?

No idea.

Definition 2.79. An ordinal β is **regular** iff $\mathrm{cf}(\beta) = \beta$. Otherwise β is called **singular**.

In particular, a regular ordinal is always a cardinal.

Lemma 2.80. Let β be an ordinal Then $\mathrm{cf}(\beta)$ is a regular cardinal, i.e.

$$\mathrm{cf}(\mathrm{cf}(\beta)) = \mathrm{cf}(\beta).$$

Proof. Suppose not. Let $f\colon \mathrm{cf}(\beta) \to \beta$ be cofinal and $g\colon \mathrm{cf}(\mathrm{cf}(\beta)) \to \mathrm{cf}(\beta)$.

Consider

$$\begin{aligned} h\colon \mathrm{cf}(\mathrm{cf}(\beta)) &\longrightarrow \beta \\ \eta &\longmapsto \sup\{f(\xi) : \xi \leqslant g(\eta)\} < \beta. \end{aligned}$$

Clearly this is cofinal. ☐

Warning 2.81. Note that in general, a composition of cofinal maps is not necessarily cofinal.

Theorem 2.82. Let $\kappa > \aleph_0$. Then κ^+ is regular.

Proof. Suppose that $\mathrm{cf}(\kappa^+) < \kappa^+$. Then $\mathrm{cf}(\kappa^+) \leqslant \kappa$, i.e. there is a cofinal function $f\colon \kappa \to \kappa^+$. By the axiom of choice, there is a function g with domain κ, such that $g(\eta)\colon \kappa \twoheadrightarrow f(\eta)$ is onto. Now define

$$\begin{aligned} h\colon \kappa \times \kappa &\longrightarrow \kappa^+ \\ (\eta, \xi) &\longmapsto g(\eta)(\xi). \end{aligned}$$

Clearly this is surjective, but $|\kappa \times \kappa| < \kappa^+$, by Theorem 2.70. ☐

- $\aleph_0, \aleph_1, \aleph_2, \dots$ are regular,
- $\aleph_\omega$ is singular,
- $\aleph_{\omega+1}, \aleph_{\omega+2}, \dots$ are regular,
- $\aleph_{\omega+\omega}$ is singular,
- $\aleph_{\omega+\omega+1}, \dots$ are regular,
- $\aleph_{\omega+\omega+\omega}$ is singular,
- ...
- $\aleph_{\omega_1}$ is singular,
- $\aleph_{\omega_1+1}, \dots$ is regular,
- $\aleph_{\omega_2}$ is singular.

Question 2.82.36 (Hausdorff)**.** Is there a regular limit cardinal?

Maybe. This is independent of ZFC, cf. Definition 2.107.

Theorem 2.83 (Hausdorff)**.**

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}.$$

Proof. Recall that

$$\aleph_{\alpha+1}^{\aleph_\beta} = |{}^{\aleph_\beta}\aleph_{\alpha+1}|.$$

- First case: $\beta \geqslant \alpha + 1$. Note that for all $\gamma \leqslant \beta$ we have
$$\aleph_\gamma^{\aleph_\beta} \leqslant \aleph_\beta^{\aleph_\beta} \leqslant \left(2^{\aleph_\beta}\right)^{\aleph_\beta} = 2^{\aleph_\beta \cdot \aleph_\beta} = 2^{\aleph_\beta} \leqslant \aleph_\gamma^{\aleph_\beta}.$$
So in this case $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$ and $\aleph_{\alpha+1}^{\aleph_\beta} = 2^{\aleph_\beta}$. Thus
$$\aleph_{\alpha+1}^{\aleph_\beta} = 2^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}.$$

- Second case: Suppose $\beta < \alpha + 1$. By case hypothesis and because $\aleph_{\alpha+1}$ is regular, no $f\colon \aleph_\beta \to \aleph_{\alpha+1}$ is unbounded. So
$${}^{\aleph_\beta}\aleph_{\alpha+1} = \bigcup_{\xi < \aleph_{\alpha+1}} {}^{\aleph_\beta}\xi$$
for each $\xi < \aleph_{\alpha+1}$, $|\xi| \leqslant \aleph_\alpha$, hence
$$|{}^{\aleph_\beta}\xi| \leqslant \aleph_\alpha^{\aleph_\beta}$$
for each $\xi < \aleph_{\alpha+1}$. Therefore,
$$\aleph_{\alpha+1}^{\aleph_\beta} \leqslant \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta} \leqslant \aleph_{\alpha+1}^{\aleph_\beta}.$$

$\square$

[Lecture 13, 2023-11-30]

Remark 2.83.37 ("Constructive" approach to ω_1)**.** There are many well-orders on ω. Let W be the set of all such well-orders. For $R, S \in W$, write $R \leqslant S$ iff R is isomorphic to an initial segment of S. Consider ${}^{W}\!/_{\sim}$, where $R \sim S :\iff R \leqslant S \wedge S \leqslant R$. Define $\leqslant$ on ${}^{W}\!/_{\sim}$ by $[R] \leqslant [S] :\iff R \leqslant S$. Clearly this is well-defined and $<$ is a well-order on ${}^{W}\!/_{\sim}$: Suppose that $\{R_n : n \in \omega\} \subseteq W$ is such that $R_{n+1} < R_n$. Then there exist $n_i \in \omega$ such that $R_i \cong R_0|_{\{x : x <_{R_0} n_i\}}$ and these form a $<_{R_0}$ strictly decreasing sequence.

So $({}^{W}\!/_{\sim})$ is a well-ordered set. Every well-order on a countable set is isomorphic to (ω, R) for some $[R] \in {}^{W}\!/_{\sim}$.

Moreover if $R \in W$, then

$$(\omega; R) \cong (\underbrace{\{[S] \in {}^{W}\!/_{\sim} : [S] < [R]\}}_{I}; <|_I),$$

where the isomorphism is given by

$$n \longmapsto [R|_{\{m : (m,n) \in R\}}].$$

This also shows that every $[R] \in {}^{W}\!/_{\sim}$ has only countably many $<$-predecessors. This then also shows that $({}^{W}\!/_{\sim}, <)$ itself is not a well-order on a countable set.

Thus $\operatorname{otp}({}^{W}\!/_{\sim}, <) = \omega_1$.

Notation 2.83.38. Let $I \neq \varnothing$ and let $\{\kappa_i : i \in I\}$ be a set of cardinals.

Then

$$\sum_{i \in I} \kappa_i := \left| \bigcup_{i \in I} (\kappa_i \times \{i\}) \right|$$

and

$$\prod_{i \in I} := \left| \bigtimes_{i \in I} \kappa_i \right|,$$

where

$$\bigtimes_{i \in I} A_i := \{f : f \text{ is a function}, \operatorname{dom}(f) = I, \forall i.\ f(i) \in A_i\}.$$

Remark 2.83.39. (C) is equivalent to $\forall i \in I.\ A_i \neq \varnothing \implies \bigtimes_{i \in I} A_i \neq \varnothing$.

Theorem 2.84 (Kőnig)**.** Let $I \neq \varnothing$. Let $\{\kappa_i : i \in I\}$, $\{\lambda_i : i \in I\}$ be sets of cardinals such that $\kappa_i < \lambda_i$ for all $i \in I$.

Then
$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i.$$

Proof. Consider a function $F\colon \bigcup_{i \in I}(\kappa_i \times \{i\}) \to \bigtimes_{i \in I} \lambda_i$. We want to show that F is not surjective.

For $i \in I$, let ξ_i be the least $\xi < \lambda_i$ such that for all $\eta < \kappa_i$
$$\underbrace{F((\eta, i))(i)}_{\in \lambda_i} \neq \xi.$$

Such ξ exists, since $\kappa_i < \lambda_i$.

Let $f \in \bigtimes_{i \in I} \lambda_i$ be defined by $i \mapsto \xi_i$.

Then $f \notin \operatorname{ran}(F)$. □

Corollary 2.85. For infinite cardinals κ, it is $\operatorname{cf}(2^\kappa) > \kappa$.

Proof. If 2^κ is a successor cardinal, then $\operatorname{cf}(2^\kappa) = 2^\kappa > \kappa$, since successor cardinals are regular.

Suppose $\operatorname{cf}(2^\kappa) \leqslant \kappa$ is a limit cardinal. Then there is some cofinal $f\colon \kappa \to 2^\kappa$. Write $\kappa_i = f(i)$ (replacing $f(i)$ by $|f(i)|^+$ we may assume that every κ_i is a cardinal).

For $i \in \kappa$, write $\lambda_i = 2^\kappa$. By Kőnig's Theorem (2.84),
$$\sup\{\kappa_i : i < \kappa\} \leqslant \sum_{i \in \kappa} \kappa_i < \prod_{i \in \kappa} \lambda_i = (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa$$

and f is not cofinal. □

Fact 2.85.40. Properties of the function $\kappa \mapsto 2^\kappa$.

- $\mu < \kappa \implies 2^\mu \leqslant 2^\kappa$ (it is independent of ZFC whether or not this is strictly increasing).
- $\operatorname{cf}(2^\kappa) \geqslant \kappa^+$.

This is "all" you can prove in ZFC.

The next goal is to show the following: (However the method might be more interesting than the result)

Theorem 2.86 (Silver)**.** If $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

Relevant concepts to prove this theorem:

Definition 2.87. Let α be a limit ordinal.

- We say that $A \subseteq \alpha$ is **unbounded** (in α), iff for all $\beta < \alpha$, there is some $\gamma \in A$ such that $\beta < \gamma$.
- We say that $A \subseteq \alpha$ is **closed**, iff it is closed with respect to the order topology on α, i.e. for all $\beta < \alpha$,

$$\sup(A \cap \beta) \in A \cup \{0\}.$$

- A is **club** (*cl*osed *un*bounded) iff it is closed and unbounded.

The interesting case is that α is a regular uncountable cardinal.

Fact 2.87.41. $A \subseteq \alpha$ being unbounded is equivalent to $f\colon \beta \to \alpha$ being cofinal, where $(\beta, \in) \overset{f}{\cong} (A, \in)$.

[Lecture 14, 2023-12-04]

Abuse of Notation 2.87.42. Sometimes we say club instead of club in κ.

Example 2.88. Let κ be a regular uncountable cardinal.

- κ is a club in κ.
- $\{\xi + 1 : \xi < \kappa\}$ is unbounded in κ, but not closed.
- For each $\alpha < \kappa$, the set $\alpha + 1 = \{\xi : \xi \leqslant \alpha\}$ is closed but not unbounded in κ.
- $\{\xi < \kappa : \xi \text{ is a limit ordinal}\}$ is club in κ.

Lemma 2.89. Let κ be regular and uncountable. Let $\alpha < \kappa$ and let $\langle C_\beta : \beta < \alpha \rangle$ be a sequence of subsets of κ which are all club in κ. Then

$$\bigcap_{\beta < \alpha} C_\beta$$

is club in κ.

Warning 2.90. This is false for $\alpha = \kappa$: Let $C_\beta := \{\xi : \xi > \beta\}$. Clearly this is club but $\bigcap_{\beta<\kappa} C_\beta = \emptyset$.

Proof of Lemma 2.89. First let $\alpha = 2$. Let $C, D \subseteq \kappa$ be club. $C \cap D$ is trivially closed:

Let $\beta < \kappa$. Suppose that $(C \cap D) \cap \beta$ is unbounded in β, so $C \cap \beta$ and $D \cap \beta$ are both unbounded in β, so $\beta \in C \cap D$.

$C \cap D$ is unbounded:

Take some $\gamma < \kappa$. Let $\gamma_0 = \gamma$ and inductively define γ_n : If n is even, let $\gamma_n := \min C \backslash (\gamma_{n-1} + 1)$, otherwise $\gamma_n := \min D \backslash (\gamma_{n-1} + 1)$.

Let $\xi = \sup\{\gamma_n : n < \omega\}$. Then $\xi = \sup\{\gamma_{2n+2} : n < \omega\} \in D$ and $\xi \in C$ by the same argument, so $\xi \in C \cap D$ (here it is important, that $\mathrm{cf}(\kappa) > \omega$) and $\xi > \gamma$.

The case $\alpha > 2$ is similar: The intersection is closed by exactly the same argument.[4]

Let's prove that $\bigcap\{C_\beta : \beta < \alpha\}$ is unbounded in κ.

We will define a sequence $\langle \gamma_i : i \leqslant \alpha \cdot \omega \rangle$[5] as follows:

Let $\gamma_0 := \gamma$. Choose

$$\gamma_{\alpha \cdot n + \beta + 1} = \min C_\beta \backslash (\gamma_{\alpha \cdot n + \beta} + 1)$$

and at limits choose the supremum.

Let $\xi = \sup_{i < \alpha \cdot \omega} \gamma_i = \sup_{i < \omega} \gamma_{\alpha \cdot n + \beta + 1} \in \bigcap_{\beta < \alpha} C_\beta$, where we have used that. $\mathrm{cf}(\kappa) > \alpha \cdot \omega$. $\square$

Definition 2.91. $F \subseteq \mathcal{P}(a)$ is a **filter** iff

(a) $X, Y \in F \implies X \cap Y \in F$,

(b) $X \in F \wedge X \subseteq Y \subseteq \kappa \implies Y \in F$,

(c) $\emptyset \notin F$,[a] $\kappa \in F$.

Let $\alpha \leqslant \kappa$. We call F **$< \alpha$-closed** iff for all $\gamma < \alpha$ and $\{X_\beta : \beta < \gamma\} \subseteq F$ then $\bigcap\{X_\beta : \beta < \gamma\} \in F$.

[a] Some authors don't require $\emptyset \notin F$, but that is a degenerate case anyway, since $\emptyset \in F \iff F = \mathcal{P}(a)$.

Intuitively, a filter is a collection of "big" subsets of a.

[4] "It is even more closed."

[5] Ordinal multiplication, i.e. $\alpha \cdot \omega = \sup_{n<\omega} \underbrace{\alpha + \ldots + \alpha}_{n \text{ times}}$.

Definition 2.92. Let κ be regular and uncountable. The **club filter** is defined as

$$\mathcal{F}_\kappa := \{X \subseteq \kappa : \exists \text{ club } C \subseteq \kappa.\ C \subseteq X\}.$$

Clearly this is a filter.

We have shown (assuming (C) to choose contained clubs):

Theorem 2.93. If κ is regular and uncountable. Then $\mathcal{F}_\kappa$ is a $< \kappa$-closed filter.

Proof. Clearly $\emptyset \notin \mathcal{F}_\kappa$, $\kappa \in \mathcal{F}_\kappa$, and $A \in \mathcal{F}_\kappa, A \subseteq B \in \kappa \implies B \in \mathcal{F}_\kappa$. In Lemma 2.89 showed that the intersection of $< \kappa$ many clubs is club. □

Definition 2.94. Let $\langle A_\beta : \beta < \alpha \rangle$ be a sequence of sets. The **diagonal intersection**, is defined to be

$$\bigtriangleup_{\beta<\alpha} A_\beta := \{\xi < \alpha : \xi \in \bigcap\{A_\beta : \beta < \xi\}\} = \bigcap_{\beta<\alpha} ([0, \beta] \cup A_\beta)$$

Remark† 2.94.43. Note that if A is closed, so is $[0, \alpha] \cup A$. Since the intersection of arbitrarily many closed sets is closed, we get that the diagonal intersection of closed sets is closed.

Lemma 2.95. Let κ be a regular, uncountable cardinal. If $\langle C_\beta : \beta < \kappa \rangle$ is a sequence of club subsets of κ, then $\bigtriangleup_{\beta<\kappa} C_\beta$ contains a club.

Proof of Lemma 2.95. Let us fix $\langle C_\beta : \beta < \alpha \rangle$. Write $D_\beta := \bigcap\{C_\gamma : \gamma \leqslant \beta\}$ for $\beta < \kappa$. Each D_β is a club, $D_\beta \subseteq C_\beta$ and $D_\beta \supseteq D_{\beta'}$ for $\beta \leqslant \beta' < \kappa$.

It suffices to show that $\bigtriangleup_{\beta<\kappa} D_\beta$ contains a club.

Claim 2.95.1. *$\bigtriangleup_{\beta<\kappa} D_\beta$ is closed in κ.*

Subproof. Cf. Remark† 2.94.43.

■

Claim 2.95.2. *$\bigtriangleup_{\beta<\kappa} D_\beta$ is unbounded in κ.*

Subproof. Fix $\gamma < \kappa$. We need to find $\delta > \gamma$ with $\delta \in \bigtriangleup_{\beta<\kappa} D_\beta$.

Define $\langle \gamma_n : n < \omega \rangle$ as follows: $\gamma_0 := \gamma$ and

$$\gamma_{n+1} := \min D_{\gamma_n} \backslash (\gamma_n + 1)$$

We have $\delta := \sup_{n<\omega} \gamma_n \in \kappa$ by cofinality of κ.

We need to show that $\delta \in D_{\overline{\gamma}}$ for all $\overline{\gamma} < \delta$.

If $\overline{\gamma} < \delta$, then $\overline{\gamma} \leqslant \gamma_n$ for some $n < \omega$. For $m \geqslant n$, $\gamma_{m+1} \in D_{\gamma_m} \subseteq D_{\gamma_n} \subseteq D_{\overline{\gamma}}$. So $D_{\overline{\gamma}} \cap \delta$ is unbounded in δ, hence $\delta \in D_{\overline{\gamma}}$. ■

□

Remark† 2.95.44. $\triangle_{\beta<\kappa} C_\beta$ actually *is* a club, since $\triangle_{\beta<\kappa} C_\beta$ is closed, again cf. Remark† 2.94.43.

Definition 2.96. Let κ be regular and uncountable. $S \subseteq \kappa$ is called **stationary** (in κ) iff $C \cap S \neq \varnothing$ for every club $C \subseteq \kappa$.

Remark† 2.96.45 (`https://mathoverflow.net/q/37503`)**.** Informally, club sets and stationary sets can be viewed as large sets of a measure space of measure 1. Clubs behave similarly to sets of measure 1 and stationary sets are analogous to sets of positive measure:

- Every club is stationary,
- the intersection of two clubs is a club,
- the intersection of a club and a stationary set is stationary,
- there exist disjoint stationary sets.

Example 2.97.

- Every $D \subseteq \kappa$ which is club in κ is stationary in κ.
- There exist disjoint stationary sets:[a] Let $\kappa = \omega_2$. Let $S_0 := \{\xi < \kappa : \mathrm{cf}(\xi) = \omega\}$ and $S_1 := \{\xi < \kappa : \mathrm{cf}(\xi) = \omega_1\}$. Clearly these are disjoint. They are both stationary: Let $C \subseteq \kappa$ be a club. Let $(\xi_i : i \leqslant \omega_1)$ be defined as follows: $\xi_0 := \min C$, $\xi_i := \min(C \setminus \sup_{j<i} \xi_j)$. For $i \leqslant \omega_1$ we have that $\xi_i = \sup_{j<i} \xi_j$. In particular $\xi_\omega \in S_0 \cap C$ and $\xi_{\omega_1} \in S_1 \cap C$.

[a]Note that clubs can never be disjoint, since their intersection is a club.

We will show later that if κ is a regular uncountable cardinal, then every stationary $S \subseteq \kappa$ can be written as $S = \bigcup_{i<\kappa} S_i$, where the S_i are stationary and pairwise disjoint.

[Lecture 15, 2023-12-07]

Theorem 2.98 (Fodor)**.** Let κ be a regular and uncountable cardinal. Let $S \subseteq \kappa$ be stationary and let $f\colon S \to \kappa$ be **regressive** in the following sense:

$f(\alpha) < \alpha$ for all $\alpha \in S$.

Then there exists a stationary subset $T \subseteq S$ and some $\nu < \kappa$ such that $f(\alpha) = \nu$ for all $\alpha \in T$.

Proof. Let S, f be given. For $\nu < \kappa$ set $S_\nu := \{\alpha \in S : f(\alpha) = \nu\}$. We aim to show that one of the S_ν is stationary. Suppose otherwise. Then for every ν there exists a club C_ν such that $S_\nu \cap C_\nu = \varnothing$.[6] Let $C = \Delta_{\nu<\kappa} C_\nu$. By Lemma 2.95 C is a club. So we may pick some $\alpha \in C \cap S$. In particular $\alpha \in C_\nu$ for all $\nu < \alpha$. Hence $f(\alpha) \neq \nu$ for all $\nu < \alpha$, so $f(\alpha) \geqslant \alpha$. But f is regressive ↯ □

2.7 Some model theory and a second proof of Fodor's Theorem

Recall the following:

Definition 2.99. A substructure $X \subseteq V_\theta$ is an **elementary substructure** of V_θ, denoted $X \prec V_\theta$,[a] iff for all formulae φ of the language of set theory and for all $x_1, \ldots, x_k \in X$,

$$(X; \in|_X) \models \varphi(x_1, \ldots, x_k) \iff (V_\theta; \in|_{V_\theta}) \models \varphi(x_1, \ldots, x_k).$$

[a]more formally $(X, \in) \prec (V_\theta)$

make this more general. Explain why V_θ is a model

Remark 2.99.46. Löwenheim-Skolem allows us to find elementary substructures of arbitrary sizes. How do we do this? Let φ be a formula. A **Skolem-function** over V_θ for φ is a function

$$f\colon {}^kV_\theta \to V_\theta,$$

where k is the number of free variables of $\exists v.\ \varphi$ and for all $x_1, \ldots, x_k \in V_\theta$, if $(V_\theta, \in) \models \exists v.\ \varphi(v, x_1, \ldots, x_k)$ then $(V_\theta, \in) \models \varphi(f(x_1, \ldots, x_k), x_1, \ldots, x_k)$.

Using (C) such Skolem-functions can be easily found for all formulae.

There is a sufficient criterion for $X \subseteq V_\theta$ to be an elementary substructure of V_θ.

Lemma 2.100 (Tarski-Vaught Test)**.** Let $X \subseteq V_\theta$. For each formula φ, let f_φ be a Skolem function over V_θ for φ. If for every φ and for all $x_1, \ldots, x_k \in X$ (where k is the number of free variables of $\exists v.\ \varphi$) $f_\varphi(x_1, \ldots, x_k) \in X$, then $X \prec V_\theta$.

Let's do a second proof of Fodor's Theorem (2.98).

[6]Here we use (C) to choose the C_ν uniformly.

Proof of Theorem 2.98. Fix $\theta > \kappa$ and look at V_θ.

Fix $S \subseteq \kappa$ stationary and $f : S \to \kappa$ regressive.

For each formula φ fix a Skolem function f_φ over V_θ for φ. Let $(X_\xi : \xi \leqslant \kappa)$ be a sequence of elementary substructures of V_θ defined as follows: Let X_0 be the least X such that $S, f \in X$ and X is closed under f_φ. Note that X_0 is countable.

For $\xi < \kappa$ let $X_{\xi+1}$ be the least $X \subseteq V_\theta$ such that $X_\xi \subseteq X$, $\min(\kappa \setminus X_\xi) \in X$ and X is closed under all f_φ. For limits $\lambda \leqslant \kappa$ let

$$X_\lambda := \bigcup_{\xi < \lambda} X_\xi.$$

Note that $|X_\xi| = |X_{\xi+1}|$ but the size may increase at limits. It is easy to see inductively that $|X_\xi| < \kappa$ for every $\xi < \kappa$, while $X_\xi \subsetneq X_{\xi'}$ for all $\xi < \xi' \leqslant \kappa$.

Also $\xi \subseteq X_\xi$ for all $\xi \leqslant \kappa$.

Claim 2.98.1. *There is a club $C \subseteq \kappa$ such that $X_\xi \cap \kappa = \xi$ for all $\xi \in C$.*

Proof of Claim 2.98.1. Write $C = \{\xi < \kappa : X_\xi \cap \kappa = \xi\}$. Trivially C is closed. Let us show that C is unbounded in κ. Let $\zeta < \kappa$. Let us define a strictly increasing sequence $\langle \xi_n : n < \omega \rangle$ as follows. Set $\xi_0 := \zeta$. Suppose ξ_n has been chosen. Look at $X_{\xi_n} \cap \kappa$. Since $|X_{\xi_n} \cap \kappa| < \kappa$, $\sup(X_{\xi_n} \cap \kappa) < \kappa$. Set $\xi_{n+1} := \sup(X_{\xi_n} \cap \kappa) + 1$. Set $\xi := \sup_{n<\omega} \xi_n$. Clearly $\zeta < \xi$.

Claim 2.98.1.1. *$\xi \in C$, i.e. $X_\xi \cap \kappa = \xi$.*

Proof of Claim 2.98.1.1. If $\eta < \xi$, then $\eta < \xi_n$ for some n and then $\eta \in \xi_n \subseteq X_{\xi_n} \subseteq X_\xi$.

Now let $\eta \in X_\xi \cap \kappa$. Then $\eta \in X_{\xi_n}$ for some $n < \omega$, so $\eta < \xi_{n+1} < \xi$, hence $X_\xi \cap \kappa \subseteq \xi$. $\square$

$\square$

Now let $\alpha \in S \cap C$, i.e. $X_\alpha \prec V_\theta$ and $\alpha = X_\alpha \cap \kappa$. $f \in X_\alpha$ and f is regressive, so $f(\alpha) < \alpha$. Write $\nu = f(\alpha)$. Let $T = \{\xi \in S : f(\xi) = \nu\}$. We have $T \in X_\alpha$, as T is definable from $S, f, \nu \in X_\alpha$.

Claim 2.98.2. *T is stationary.*

Subproof. Otherwise there is a club $D \subseteq \kappa$ such that $D \cap T = \emptyset$, i.e.

$$V_\theta \models \exists D.\ D \text{ club in } \kappa \wedge D \cap T = \emptyset$$

hence

$$X_\alpha \models \exists D.\ D \text{ club in } \kappa \wedge D \cap T = \emptyset.$$

So there is $D \in X_\alpha$ such that

$$X_\alpha \models D \text{ is club in } \kappa \wedge D \cap T = \varnothing,$$

hence

$$V_\theta \models D \text{ is club in } \kappa \wedge D \cap T = \varnothing.$$

In other words, there is some club $D \in X_\alpha$ with $D \cap T = \varnothing$.

We have $\alpha \in T$ as $\alpha \in S$ and $f(\alpha) = \nu$. Let us show that $\alpha \in D$, which gives a contradiction. For $\alpha \in D$ it suffices to show that $D \cap \alpha$ is unbounded in α. Let $\xi < \alpha$. As D is unbounded in κ, $\exists \eta > \xi.\ \eta \in D$, so

$$V_\theta \models \exists \eta > \xi.\ \eta \in D,$$

hence

$$X_\alpha \models \exists \eta > \xi.\ \eta \in D.$$

Hence there is some $\eta \in X_\alpha$ with $\eta \in D$. This means that $\xi < \underbrace{\eta}_{\in D} < \alpha$. ■

□

[Lecture 16, 2023-12-11]

Recall Fodor's Theorem (2.98).

Question 2.100.47. What happens if S is nonstationary?

Let $S \subseteq \kappa$ be nonstationary, κ uncounable and regular. Then there is a club $C \subseteq \kappa$ with $C \cap S = \varnothing$. Let us define $f\colon S \to \kappa$ in the following way:

If $\alpha \in S$ and $C \cap \alpha \neq \varnothing$, then $\max(C \cap \alpha) < \alpha$.

Define

$$f(\alpha) := \begin{cases} 0 & : C \cap \alpha = \varnothing, \\ \max(C \cap \alpha) & : C \cap \alpha \neq \varnothing. \end{cases}$$

For all $\alpha > 0$, we have that $f(\alpha) < \alpha$. If $\gamma \in \operatorname{ran}(f)$ then $f(\alpha) = \gamma$ implies either $\gamma = 0$ and $\alpha < \min(C)$ or $\gamma \in C$ and $\gamma < \alpha < \gamma'$ where $\gamma' = \min(C \setminus (\gamma + 1))$. Thus for all γ, there is only an interval of ordinals $\alpha \in S$ where $f(\alpha) = \gamma$.

Recall that $F \subseteq \mathcal{P}(\kappa)$ is a filter if $X, Y \in F \implies X \cap Y \in F$, $X \in F, X \subseteq Y \subseteq \kappa \implies Y \in F$ and $\varnothing \notin F, \kappa \in F$.

Definition 2.101. A filter F is an **ultrafilter** iff for all $X \subseteq \kappa$ either $X \in F$ or $\kappa \setminus X \in F$.

Example 2.102. Examples of filters:

(a) Let $\kappa \geqslant \aleph_0$ and let $F = \{X \subseteq \kappa : \kappa \setminus X \text{ is finite}\}$. This is called the

Fréchet filter or **cofinal filter**. It is not an ultrafilter (consider for example the even and odd numbers[a]).

(b) Let κ be uncountable and regular. Then $\mathcal{F}_\kappa := \{X \subseteq \kappa : \exists C \subseteq \kappa \text{ club in } \kappa . C \subseteq X\}$.

[a] we consider limit ordinals to be even

Question 2.102.48. Is $\mathcal{F}_\kappa$ an ultrafilter?

This is certainly not the case if $\kappa \geqslant \aleph_2$, because then $S_0 := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ and $S_1 := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega_1\}$ are both stationary and clearly disjoint. So neither S_0 nor $S_1 \subseteq \kappa \backslash S_0$ contains a club.

For $\kappa < \aleph_1$ this argument does not work, since there is only one cofinality.

Theorem 2.103 (Solovay)**.** Let κ be regular and uncountable. If $S \subseteq \kappa$ is stationary, there is a sequence $\langle S_i : i < \kappa \rangle$ of pairwise disjoint stationary subsets of κ such that $S = \bigcup S_i$.

Corollary 2.104. $\mathcal{F}_{\aleph_1}$ is not an ultrafilter.

Proof. Apply Solovay's Theorem (2.103) to $S = \aleph_1$. Let $\aleph_1 = A \cup B$ where A and B are both stationary and disjoint. Then use the argument from above. $\square$

Proof of Theorem 2.103. [7] We will only prove this for $\aleph_1$. Fix $S \subseteq \aleph_1$ stationary.

For each $0 < \alpha < \omega_1$, either α is a successor ordinal or α is a limit ordinal and $\mathrm{cf}(\alpha) = \omega$.

Let $S^* := \{\alpha \in S \backslash \{0\} : \alpha \text{ is a limit ordinal}\}$. S^* is still stationary: Let $C \subseteq \omega_1$ be a club, then $D = \{\alpha \in C \backslash \{0\} : \alpha \text{ is a limit ordinal}\}$ is still a club, so

$$S^* \cap C = S^* \cap D = S \cap D \neq \varnothing.$$

Let

$$\langle \langle \gamma_n^\alpha : n < \omega \rangle : \alpha \in S^* \rangle$$

be such that $\langle \gamma_n^\alpha : n < \omega \rangle$ is cofinal in α.

Claim 2.103.1. *There exists $n < \omega$ such that for all $\delta < \omega_1$ the set*

$$\{\alpha \in S^* : \gamma_n^\alpha > \delta\}$$

is stationary.

[7] "This is one of the arguments where it is certainly worth it to look at it again."

Subproof. Otherwise for all $n < \omega$, there is a δ such that $\{\alpha \in S^* : \gamma_n^\alpha > \delta\}$ is nonstationary. Let δ_n be the least such δ. Let C_n be a club disjoint from

$$\{\alpha \in S^* : \gamma_n^\alpha > \delta_n\},$$

i.e. if $\alpha \in S^* \cap C_n$, then $\gamma_n^\alpha \leqslant \delta_n$. Let $\delta^* := \sup_{n<\omega} \delta_n$.

Let $C = \bigcap_{n<\omega} C_n$. Then C is a club. We must have that if $\alpha \in S^* \cap C$ then $\gamma_n^\alpha \leqslant \delta^*$ for all n.

Let $C' := C \backslash (\delta^* + 1)$. C' is still club. As S^* is stationary, we may pick some $\alpha \in S^* \cap C'$. But then $\gamma_n^\alpha > \delta^*$ for n large enough as $\langle \gamma_n^\alpha : n < \omega \rangle$ is cofinal in α $\lightning$. ■

Let $n < \omega$ be as in Claim 2.103.1. Consider

$$\begin{aligned} f \colon S^* &\longrightarrow \omega_1 \\ \alpha &\longmapsto \gamma_n^\alpha. \end{aligned}$$

Clearly this is regressive.

We will now define a strictly increasing sequence $\langle \delta_i : i < \omega_1 \rangle$ as follows:

Let $\delta_0 = 0$.

For $0 < i < \omega_1$ suppose that $\delta_j, j < i$ have been defined. Let $\delta := (\sup_{j<i} \delta_j) + 1$. By Claim 2.103.1 (rather, by the choice of n), we have that $\{\alpha \in S^* : \gamma_n^\alpha > \delta\}$ is stationary. Hence by Fodor there is some stationary $T \subseteq S^*$ and some δ' such that for all $\alpha \in T$ we have $\gamma_n^\alpha = \delta'$.

Write $\delta_i = \delta'$ and $T_i = T$.

By construction, all the T_i are stationary. Since the δ_i are strictly increasing and since $\gamma_n^\alpha = \delta_i$ for all $\alpha \in T_i$, we have that the T_i are disjoint.

Now let

$$S_i := \begin{cases} T_i & : i > 0, \\ T_0 \cup (S \backslash \bigcup_{j>0} T_j) & : i = 0. \end{cases}$$

Then $\langle S_i : i < \omega_1 \rangle$ is as desired. □

We now want to do another application of Fodor's Theorem (2.98). Recall that $2^\kappa > \kappa$, in fact $\operatorname{cf}(2^\kappa) > \kappa$ by Kőnig's Theorem (2.84) (cf. Corollary 2.85).

Trivially, if $\kappa \leqslant \lambda$ then $2^\kappa \leqslant 2^\lambda$. This is in some sense the only thing we can prove about successor cardinals. However we can say something about singular cardinals:

Theorem 2.105 (Silver)**.** Let κ be a singular cardinal of uncountable cofinality. Assume that $2^\lambda = \lambda^+$ for all (infinite) cardinals $\lambda < \kappa$. Then $2^\kappa = \kappa^+$.

Definition 2.106. GCH, the **generalized continuum hypothesis** is the statement that $2^\lambda = \lambda^+$ holds for all infinite cardinals λ,

Recall that CH says that $2^{\aleph_0} = \aleph_1$. So GCH $\implies$ CH.

Silver's Theorem (2.105) says that if GCH is true below κ, then it is true at κ.

The proof of Silver's Theorem (2.105) is quite elementary, so we will do it now, but the statement can only be fully appreciated later.

[Lecture 17, 2023-12-14]

We now want to prove Silver's Theorem (2.105).

Remark 2.106.49. The hypothesis of Silver's Theorem (2.105) is consistent with ZFC.

We will only prove Silver's Theorem (2.105) in the special case that $\kappa = \aleph_{\omega_1}$ (see Silver's Theorem (case of $\aleph_{\omega_1}$) (2.86)). The general proof differs only in notation.

Remark 2.106.50. It is important that the cofinality is uncountable. For example it is consistent with ZFC that $2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$ but at the same time $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Proof of Theorem 2.86. We need to count the number of $X \subseteq \aleph_{\omega_1}$. Let us fix $\langle f_\lambda : \lambda < \kappa \text{ an infinite cardinal}\rangle$ such that $f_\lambda \colon \mathcal{P}(\lambda) \to \lambda^+$ is bijective for each $\lambda < \kappa$.

For $X \subseteq \aleph_{\omega_1}$ define

$$\begin{aligned} f_X \colon \omega_1 &\longrightarrow \aleph_{\omega_1} \\ \alpha &\longmapsto f_{\aleph_\alpha}(X \cap \aleph_\alpha). \end{aligned}$$

Claim 2.86.1. *For $X, Y \subseteq \aleph_{\omega_1}$ it is $X \neq Y \iff f_X \neq f_Y$.*

Subproof. $X \neq Y$ holds iff $X \cap \aleph_\alpha \neq Y \cap \aleph_\alpha$ for some $\alpha < \omega_1$. But then $f_X(\alpha) \neq f_Y(\alpha)$. ■

For $X, Y \subseteq \aleph_{\omega_1}$ write $X \leqslant Y$ iff

$$\{\alpha < \omega_1 : f_X(\alpha) \leqslant f_Y(\alpha)\}$$

is stationary.

Claim 2.86.2. *For all $X, Y \subseteq \aleph_{\omega_1}$, $X \leqslant Y$ or $Y \leqslant X$.*

Subproof. Suppose that $X \not\leqslant Y$ and $Y \not\leqslant X$. Then there are clubs $C, D \subseteq \omega_1$ such that

$$C \cap \{\alpha < \omega_1 : f_X(\alpha) \leqslant f_Y(\alpha)\} = \varnothing$$

and

$$D \cap \{\alpha < \omega_1 : f_Y(\alpha) \leqslant f_X(\alpha)\} = \varnothing.$$

Note that $C \cap D$ is a club. Take some $\alpha \in C \cap D$. But then $f_X(\alpha) \leqslant f_Y(\alpha)$ or $f_Y(\alpha) \leqslant f_X(\alpha)$ ↯ ■

Claim 2.86.3. . *Let $X \subseteq \aleph_{\omega_1}$. Then*

$$|\{Y \subseteq \aleph_{\omega_1} : Y \leqslant X\}| \leqslant \aleph_{\omega_1}.$$

Subproof. Write $A := \{Y \subseteq X_{\omega_1} : Y \leqslant X\}$. Suppose $|A| \geqslant \aleph_{\omega_1+1}$. For each $Y \in A$ we have that

$$S_Y := \{\alpha : f_Y(\alpha) \leqslant f_X(\alpha)\}$$

is a stationary subset of ω_1. Since by assumption $2^{\aleph_1} = \aleph_2$, there are at most $\aleph_2$ such S_Y.

Suppose that for each $S \subseteq \omega_1$,

$$|\{Y \in A : S_Y = S\}| < \aleph_{\omega_1+1}.$$

Then A is the union of $\leqslant \aleph_2$ many sets of size $< \aleph_{\omega_1+1}$. Thus this is a contradiction since $\aleph_{\omega_1+1}$ is regular.

So there exists a stationary $S \subseteq \omega_1$ such that

$$A_1 = \{Y \subseteq \aleph_{\omega_1} : S_Y = S\}$$

has cardinality $\aleph_{\omega_1+1}$. We have

$$f_Y(\alpha) \leqslant f_X(\alpha) = f_{\aleph_\alpha}(X \cap \aleph_\alpha) < \aleph_{\alpha+1}$$

for all $Y \in A_1, \alpha \in S$.

Let $\langle g_\alpha : \alpha \in S\rangle$ be such that $g_\alpha \colon \aleph_\alpha \twoheadrightarrow f_X(\alpha) + 1$ is a surjection for all $\alpha \in S$.

Then for each $Y \in A_1$ define

$$\begin{aligned} \overline{f}_Y \colon S &\longrightarrow \aleph_{\omega_1} \\ \alpha &\longmapsto \min\{\xi : g_\alpha(\xi) = f_Y(\alpha)\}. \end{aligned}$$

Let D be the set of all limit ordinals $< \omega_1$. Then $S \cap D$ is a stationary set: If C is a club, then $C \cap D$ is a club, hence $(S \cap D) \cap C = S \cap (D \cap C) \neq \varnothing$.

Now to each $Y \in A$ we may associate a regressive function

$$\begin{aligned} h_Y \colon S \cap D &\longrightarrow \omega_1 \\ \alpha &\longmapsto \min\{\beta < \alpha : \overline{f}_Y(\alpha) < \aleph_\beta\}. \end{aligned}$$

h_Y is regressive, so by Fodor's Theorem (2.98) there is a stationary $T_Y \subseteq S \cap D$ on which h_Y is constant.

By an argument as before, there is a stationary $T \subseteq S \cap D$ such that

$$|A_2| = \aleph_{\omega_1+1},$$

where $A_2 := \{Y \in A_1 : T_Y = T\}$.

Let $\beta < \omega_1$ be such that for all $Y \in A_2$ and for all $\alpha \in T$, $h_Y(\alpha) = \beta$. Then $\overline{f}_Y(\alpha) < \aleph_\beta$ for all $Y \in A_2$ and $\alpha \in T$.

There are at most $\aleph_\beta^{\aleph_1}$ many functions $T \to \aleph_\beta$, but

$$\begin{aligned}\aleph_\beta^{\aleph_1} &\leqslant 2^{\aleph_\beta \cdot \aleph_1} \\ &= \aleph_{\beta+1} \cdot \aleph_2 \\ &< \aleph_{\omega_1}.\end{aligned}$$

Suppose that for each function $\tilde{f}: T \to \aleph_\beta$ there are $< \aleph_{\omega_1+1}$ many $Y \in A_2$ with $\overline{f}_Y \cap T = \tilde{f}$.

Then A_2 is the union of $< \aleph_{\omega_1}$ many sets each of size $< \aleph_{\omega_1+1}$ $\lightning$. Hence for some $\tilde{f}: T \to \aleph_\beta$,

$$|A_3| = \aleph_{\omega_1+1},$$

where $A_3 = \{Y \in A_2 : \overline{f}_Y|_T = \tilde{f}\}$.

Let $Y, Y' \in A_3$ and $\alpha \in T$. Then

$$\overline{f}_Y(\alpha) = \overline{f}_{Y'}(\alpha),$$

hence

$$f_{\aleph_\alpha}(Y \cap \aleph_\alpha) = f_Y(\alpha) = f_{Y'}(\alpha) = f_{\aleph_\alpha}(Y' \cap \aleph_\alpha),$$

i.e. $Y \cap \aleph_\alpha = Y' \cap \aleph_\alpha$. Since T is cofinal in ω_1, it follows that $Y = Y'$. So $|A_3| \leqslant 1$ $\lightning$ ∎

Let us now define a sequence $\langle X_i : i < \aleph_{\omega_1+1} \rangle$ of subsets of $\aleph_{\omega_1+1}$ as follows:

Suppose $\langle X_j : j < i \rangle$ were already chosen. Consider

$$\{Y \subseteq \aleph_{\omega_1} : \exists j < i.\ Y \leqslant X_j\} = \bigcup_{j<i} \{Y \subseteq \aleph_{\omega_1} : Y \leqslant X_j\}.$$

This set has cardinality $\leqslant \aleph_{\omega_1}$ by Claim 2.86.3. Let $X_i \subseteq \aleph_{\omega_1}$ be such that $X_i \nleqslant X_j$ for all $j < i$.

The set

$$P := \{Y \subseteq \aleph_{\omega_1} : \exists i < \aleph_{\omega_1+1}.\ Y \leqslant X_i\} = \bigcup_{i<\aleph_{\omega_1+1}} \{Y \subseteq \aleph_{\omega_1} : Y \leqslant X_i\}$$

has size $\leqslant \aleph_{\omega_1+1}$ (in fact the size is exactly $\aleph_{\omega_1+1}$).

On the other hand $P = \mathcal{P}(\aleph_{\omega_1})$ because if $X \subseteq \aleph_{\omega_1}$ is such that $X \nleqslant X_i$ for all $i < \aleph_{\omega_1+1}$, then $X_i \leqslant X$ for all $i < \aleph_{\omega_1+1}$, so such a set X does not exist by Claim 2.86.3.

□

[Lecture 18, 2023-12-18]

Definition 2.107.
- A cardinal κ is called **weakly inaccessible** iff κ is uncountable,[a] regular and $\forall \lambda < \kappa.\ \lambda^+ < \kappa$.
- A cardinal κ is **(strongly) inaccessible** iff κ is uncountable, regular and $\forall \lambda < \kappa.\ 2^\lambda < \kappa$.

[a] dropping this we would get that $\aleph_0$ is inaccessible

Remark 2.107.51. Since $2^\lambda \geqslant \lambda^+$, strongly inaccessible cardinals are weakly inaccessible.

If GCH holds, the notions coincide.

Theorem 2.108. If κ is inaccessible, then $V_\kappa \models \mathsf{ZFC}$.[a]

[a] More formally $(V_\kappa, \in|_{V_\kappa}) \models \mathsf{ZFC}$.

Proof. Since κ is regular, (Rep) works. Since $2^\lambda < \kappa$, (Pow) works. The other axioms are trivial. □

Corollary 2.109. ZFC does not prove the existence of inaccessible cardinals, unless ZFC is inconsistent.

Proof. If ZFC is consistent, it can not prove that it is consistent. In particular, it can not prove the existence of a model of ZFC. □

Definition 2.110 (Ulam)**.** A cardinal $\kappa > \aleph_0$ is **measurable** iff there is an ultrafilter U on κ, such that U is not principal[a] and $< \kappa$-closed,i.e. if $\theta < \kappa$ and $\{X_i : i < \theta\} \subseteq U$, then $\bigcap_{i<\theta} X_i \in U$.

[a] i.e. $\{\xi\} \notin U$ for all $\xi < \kappa$

Goal. *We want to prove that if κ is measurable, then κ is inaccessible and there are κ many inaccessible cardinals below κ (i.e. κ is the κ^{th} inaccessible).*

Theorem 2.111. The following are equivalent:

1. κ is a measurable cardinal.

2. There is an elementary embedding[a] $j\colon V \to M$ with M transitive such that $j|_\kappa = \mathrm{id}$, $j(\kappa) \neq \kappa$.

[a]Recall: $j\colon V \to M$ is an **elementary embedding** iff $j''V = \{j(x) : x \in V\} \prec M$, i.e. for all formulae φ and $x_1, \dots, x_k \in V$, $V \models \varphi(x_1, \dots, x_u) \iff M \models \varphi(j(x_1), \dots, j(x_u))$.

Proof. 2. $\implies$ 1.: Fix $j\colon V \to M$. Let $U = \{X \subseteq \kappa : \kappa \in j(X)\}$. We need to show that U is an ultrafilter:

- Let $X, Y \in U$. Then $\kappa \in j(X) \cap j(Y)$. We have $M \models j(X \cap Y) = j(X) \cap j(Y)$, and thus $j(X \cap Y) = j(X) \cap j(Y)$. It follows that $X \cap Y \in U$.
- Let $X \in U$ and $X \subseteq Y \subseteq \kappa$. Then $\kappa \in j(X) \subseteq j(Y)$ by the same argument, so $Y \in U$.
- We have $j(\varnothing) = \varnothing$ (again $M \models j(\varnothing)$ is empty), hence $\varnothing \notin U$.
- $\kappa \in U$ follows from $\kappa \in j(\kappa)$. This is shown as follows:

  Claim 1. *For every ordinal α, $j(\alpha)$ is an ordinal such that $j(\alpha) \geqslant \alpha$.*

  Subproof. $\alpha \in \mathrm{OR}$ can be written as

  $$\forall x \in \alpha.\ \forall y \in x.\ y \in \alpha \wedge \forall x \in \alpha.\ \forall y \in \alpha.\ (x \in y \vee x = y \vee y \in x).$$

  So if α is an ordinal, then $M \models$ "$j(\alpha)$ is an ordinal" in the sense above. Therefore $j(\alpha)$ really is an ordinal.

  If the claim fails, we can pick the least α such that $j(\alpha) < \alpha$. Then $M \models j(j(\alpha)) < j(\alpha)$, i.e. $j(j(\alpha)) < j(\alpha)$ contradicting the minimality of α. ∎

  Therefore as $j(\kappa) \neq \kappa$, we have $j(\kappa) > \kappa$, i.e. $\kappa \in j(\kappa)$.

- U is an ultrafilter: Let $X \subseteq \kappa$. Then $\kappa \in j(\kappa) = j(X \cup (\kappa \setminus X)) = j(X) \cup j(\kappa \setminus X)$. So $X \in U$ or $\kappa \setminus X \in U$.

  Let $\theta < \kappa$ and $\{X_i : i < \theta\} \subseteq U$. Then $\kappa \in j(X_i)$ for all $i < \theta$, hence

  $$\kappa \in \bigcap_{i<\theta} j(X_i) = j\left(\bigcap_{i<\theta} X_i\right) \in U.$$

  This holds since $j(\theta) = \theta$ (as $\theta < \kappa$), so $j(\langle X_i : i < \theta \rangle) = \langle j(X_i) : i < \theta \rangle$.

  Also if $\xi < \kappa$, then $j(\{\xi\}) = \{\xi\}$ so $\kappa \notin j(\{\xi\})$ and $\{\xi\} \notin U$.

1. $\implies$ 2. Fix U. Let ${}^\kappa V$ be the class of all function from κ to V. For $f, g \in {}^\kappa V$ define $f \sim g :\iff \{\xi < \kappa : f(\xi) = g(\xi)\} \in U$. This is an equivalence relation since U is a filter. Write $[f] = \{g \in {}^\kappa V : g \sim f \wedge$

$g \in V_\alpha$ for the least α such that there is some $h \in V_\alpha$ with $h \sim f\}$.[8] For any two such equivalence classes $[f], [g]$ define

$$[f] \tilde{\in} [g] :\iff \{\xi < \kappa : f(\xi) \in g(\xi)\} \in U.$$

This is independent of the choice of the representatives, so it is well-defined. Now write $\mathcal{F} = \{[f] : f \in {}^\kappa V\}$ and look at $(\mathcal{F}, \tilde{\in})$.

The key to the construction is Łoś's Theorem (2.112) (see below). Given Łoś's Theorem (2.112), we may define an elementary embedding $\overline{j} \colon (V, \in) \to (\mathcal{F}, \tilde{\in})$ as follows:

Let $\overline{j}(x) = [c_x]$, where $c_x \colon \kappa \to \{x\}$ is the constant function with value x.

Then

$$\begin{aligned} (V, \in) \models \varphi(x_1, \ldots, x_k) &\iff \{\xi < \kappa : (V, \in) \models \varphi(c_{x_1}(\alpha), \ldots, c_{x_k}(\alpha))\} \in U \\ &\overset{\text{Łoś}}{\iff} (\mathcal{F}, \tilde{\in}) \models \varphi(\overline{j}(x_1), \ldots, \overline{j}(x_k)). \end{aligned}$$

Let us show that $(\mathcal{F}, \tilde{\in})$ is well-founded. Otherwise there is $\langle f_n : n < \omega \rangle$ such that $f_n \in {}^\kappa V$ and $[f_{n+1}] \tilde{\in} [f_n]$ for all $n < \omega$.

Then $X_n := \{\xi < \kappa : f_{n+1}(\xi) \in f_n(\xi)\} \in U$, so $\bigcap X_n \in U$. Let $\xi_0 \in \bigcap X_n$. Then $f_0(\xi_0) \ni f_1(\xi_0) \ni f_2(\xi_0) \ni \ldots \lightning$.

Note that $\tilde{\in}$ is set-like, therefore by the Mostowski Collapse (2.61) there is some transitive M with $(\mathcal{F}, \tilde{\in}) \overset{\sigma}{\cong} (M, \in)$.

We can now define an elementary embedding $j \colon V \to M$ by $j := \sigma \circ \overline{j}$.

It remains to show that $\alpha < \kappa \implies j(\alpha) = \alpha$. This can be done by induction: Fix α. We already know $j(\alpha) \geqslant \alpha$. Suppose $\beta \in j(\alpha)$. Then $\beta = \sigma([f])$ for some f and $\sigma([f]) \in \sigma([c_\alpha])$, i.e. $[f] \tilde{\in} [c_\alpha]$. Thus $\{\xi < \kappa : f(\xi) \in \underbrace{c_\alpha(\xi)}_{\alpha}\} \in U$.

Hence there is some $\delta < \alpha$ such that

$$X_\delta := \{\xi < \kappa : f(\xi) = \delta\} \in U,$$

as otherwise $\forall \delta < \alpha.\ \kappa \setminus X_\delta \in U$, i.e. $\varnothing = (\bigcap_{\delta < \alpha} \kappa \setminus X_\delta) \cap X \in U \lightning$. We get $[f] = [c_\delta]$, so $\beta = \sigma([f]) = \sigma([c_\delta]) = j(\delta) = \delta$, where for the last equality we have applied the induction hypothesis. So $j(\alpha) \leqslant \alpha$.

For all $\eta < \kappa$, we have $\eta = \sigma([c_\eta]) < \sigma([c_{\mathrm{id}}]) < \sigma([c_\kappa])$, so $j(\kappa) > \kappa$.

□

[8] This is know as **Scott's Trick**. Note that by defining equivalence classes in the usual way (i.e. without this trick), one ends up with proper classes: For $f \colon \kappa \to V$, we can for example change $f(0)$ to be an arbitrary V_α and get another element of $[f]$.

Theorem 2.112 (Łoś)**.** For all formulae φ and for all $f_1, \ldots, f_k \in {}^{\kappa}V$,

$$(\mathcal{F}, \tilde{\in}) \models \varphi([f_1], \ldots, [f_k]) \iff \{\xi < \kappa : (V, \in) \models \varphi(f_1(\xi), \ldots, f_k(\xi))\} \in U.$$

Proof. Induction on the complexity of φ. □

[Lecture 19, 2024-01-11]

Beginning with this lecture, the material is no longer relevant for the exam.

Recall that $\exists x \in y.\ \varphi$ abbreviates $\exists x.\ x \in y \wedge \varphi$ and $\forall x \in y.\ \varphi$ abbreviates $\forall x.\ x \in y \to \varphi$.

Definition 2.113 (Arithmetical Hierarchy)**.** Let φ be a $\mathcal{L}_{\in}$-formula. We say that φ is Δ_0 (or Σ_0 or Π_0) iff it is in the smallest set Γ of formulas such that

(1) Γ contains all **atomic** formulas ($x \in y$,$x = y$).

(2) If $\varphi, \psi \in \Gamma$, then so are $\neg\varphi$ and $\varphi \wedge \psi$.[a]

(3) If $\varphi \in \Gamma$, then $(\exists x \in y.\ \varphi), (\forall x \in y.\ \varphi) \in \Gamma$.

If $\varphi(x_0, \ldots, x_m) \in \Sigma_n$, then $(\forall x_0.\ \ldots \forall x_m.\ \varphi(x_0, \ldots, x_m)) \in \Pi_{n+1}$. If $\mathsf{ZFC} \models \varphi \leftrightarrow \psi$ and $\varphi \in \Sigma_n$, then $\psi \in \Sigma_n$.

If $\varphi(x_0, \ldots, x_m) \in \Pi_n$, then $(\exists x_0.\ \ldots \exists x_m.\ \varphi(x_0, \ldots, x_m) \in \Sigma_{n+1}$. If $\mathsf{ZFC} \models \varphi \leftrightarrow \psi$ and $\varphi \in \Pi_n$, then $\psi \in \Pi_n$.

$\Delta_n := \Sigma_n \cap \Pi_n$.

[a] It follows that $\varphi \vee \psi$, $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$ are also in Γ.

Notation 2.113.52. Assume that M is transitive and φ is sentence. Then

$$M \models \varphi$$

means that $(M, \in|_M) \models \varphi$.

If $a_0, \ldots, a_n \in M$ and $\varphi(x_0, \ldots, x_n)$ is a $\mathcal{L}_{\in}$-formula, then we say $M \models \varphi(a_0, \ldots, a_n)$ iff M satisfies $\varphi(x_0, \ldots, x_n)$ for the assignment $x_i \mapsto a_i$.

Lemma 2.114. Let M be transitive, $\varphi \in \Delta_0$ and $a_0, \ldots, a_n \in M$. Then $M \models \varphi(a_0, \ldots, a_n)$ iff $V \models \varphi(a_0, \ldots, a_n)$.

Proof. Clearly $M \models a_i \in a_j \iff V \models a_i \in a_j$ and $M \models a_i = a_j \iff V \models a_i = a_j$, i.e. the lemma holds for atomic φ.

It is clear that if $M \models \varphi_i \iff V \models \varphi_i, i = 1, 2$, then also $M \models \neg\varphi_i \iff V \models \neg\varphi_i$ and $M \models \varphi_1 \wedge \varphi_2 \iff V \models \varphi_1 \wedge \varphi_2$.

Assume that the lemma holds for φ. Then it also holds for $\exists a_i \in a_j.\ \varphi$: We have that $a_i \in a_j$ is atomic and by the assumption that the lemma holds for φ so since M is transitive, a witness can be transferred from V to M and vice versa. The case of $\forall a_i \in a_j.\ \varphi$ can be treated similarly. $\square$

A similar arguments yields **upwards absoluteness** for Σ_1-formulas and **downwards absoluteness** for Π_1-formulas:

Lemma 2.115. Let M be transitive. Let $\varphi(x_0, \ldots, x_n) \in \mathcal{L}_\in$ and $a_0, \ldots, a_n \in M$. Then

- If φ is Σ_1, then
$$M \models \varphi(a_0, \ldots, a_n) \implies V \models \varphi(a_0, \ldots, a_n).$$
- If φ is Π_1, then
$$V \models \varphi(a_0, \ldots, a_n) \implies M \models \varphi(a_0, \ldots, a_n).$$

Definition 2.116. Assume that T is a theory and $\varphi \in \mathcal{L}_\in$ a formula We say that φ is Δ_1^T iff there are formulas ψ, τ such that $\psi \in \Sigma_1$, $\tau \in \Pi_1$ and
$$T \vdash \varphi \leftrightarrow \psi \leftrightarrow \tau.$$

Again by a similar argument we get:

Lemma 2.117. Let M be a transitive model of a theory T. Let φ be a Δ_1^T formula and $a_0, \ldots, a_n \in M$. Then $M \models \varphi(a_0, \ldots, a_n) \iff V \models \varphi(a_0, \ldots, a_n)$.

Lemma 2.118. Let φ denote the statement "R is a well-founded relation". Then $\varphi \in \Delta_1^{\mathsf{ZFC}^-}$.

Proof. φ is equivalent to

- R is a relation (Δ_0) and
- $\forall b.\ b \cap \operatorname{ran}(R) = \varnothing \vee \exists x \in b.$ "x is R-minimal".

We only need to care about the second point. This is equivalent (using (C)!) to the statement that there is no
$$f\colon \omega \to \operatorname{dom}(R) \cup \operatorname{ran}(R) \text{ such that } \forall n < \omega.\ f(n+1) R f(n),$$
which can be written as a Π_1-formula. With the help of ranks, we can also write it as a Σ_1-formula:

$\exists r\colon \mathrm{OR} \to \operatorname{dom}(R) \cup \operatorname{ran}(R).\ \forall x \in \operatorname{dom}(R) \cup \operatorname{ran}(R).\ r(x) = \{\sup(r(y)+1) : yRx\}.$

So $\varphi \in \Delta_1^{\mathsf{ZFC}^-}$. $\square$

Lemma 2.119. Assume that M is transitive. Then

(1) $M \models$ (Ext).

(2) $M \models$ (Fund).

(3) If $\omega \in M$, then $M \models$ (Inf).

(4) If M is closed under $(x, y) \mapsto \{x, y\}$, then $M \models$ (Pair).

(5) If M is closed under $x \mapsto \bigcup x$, then $M \models$ (Union).

Proof. (1) Let $x, y \in M$ such that $M \models \forall t.\ t \in x \iff t \in y$. Since M is transitive $V \models \forall t.\ t \in x \iff t \in y$. Since $V \models$ Ext, we can apply $V \models x = y \iff M \models x = y$.

(2) We need to show $M \models \forall y \neq \varnothing.\ \exists x \in y.\ x \cap y = \varnothing$. Let $y \in M$. Since $V \models$ (Fund), $V \models \exists x \in y.\ x \cap y = \varnothing$. Note that this is a Δ_0-formula, hence $M \models \exists x \in y.\ x \cap y = \varnothing$.

(3) By assumption $\omega \in M$. Since M is transitive, we get $\omega \subseteq M$. Hence ω is a witness for (Inf).

(4) Trivial.

(5) Trivial.

$\square$

3 Forcing

Recall that a structure $\mathbb{P} = (P, \leqslant)$ is a partially ordered set (**poset**) if $\leqslant$ is reflexive, symmetric and transitive.

Definition 3.1. A non-empty poset $\mathbb{P} = (P, \leqslant)$ is called a **forcing notion**. The elements of P are called **conditions**. If $q \leqslant p$ we say that q is **stronger** than p.[a] $D \subseteq P$ is called **dense** iff $\forall p \in P.\ \exists q \in D.\ q \leqslant p$.

Let $p \in P, D \subseteq P$. Then D is **dense below p** iff $\forall P \ni q \leqslant p.\ \exists r \in D.\ r \leqslant q$.

$G \subseteq P$ is called a **filter** iff

(1) $\forall p, q \in G.\ \exists r \in G.\ r \leqslant p \wedge r \leqslant q$.

(2) $(p \in G \wedge p \leqslant q) \implies q \in G$.

For $p, q \in P$ we say that p and q are **compatible**, $p || q$, iff $\exists r \in P.\ r \leqslant p \wedge r \leqslant q$. Otherwise they are **incompatible**, $p \perp q$.

Let $\mathcal{D}$ be a family of dense subsets of P and G a filter. We say that G is $\mathcal{D}$-generic iff $\forall D \in \mathcal{D}.\ G \cap D \neq \emptyset$.

[a] i.e. it carries more information.

Lemma 3.2. Let $\mathcal{P} = (P, \leqslant)$ be a poset, $\mathcal{D}$ a countable family of dense subsets of P and $p \in P$. Then there exists a $\mathcal{D}$-generic filter $G \subseteq P$ such that $p \in G$.

Proof. Fix p as above. Let $\langle D_n : n < \omega \rangle$ be an enumeration of $\mathcal{D}$. Let $p_0 \leqslant p$ be such that $p_0 \in D_0$. If p_n is given, let $p_{n+1} \leqslant p_n$ be such that $p_{n+1} \in D_{n+1}$. This is possible since $\mathcal{D}$ is a collection of dense sets. Define $G := \{q \in P : \exists n.\ p_n \leqslant q\}$.

G is a filter: Let $r, q \in G$. Let $n_r, n_q < \omega$ such that $p_{n_r} \leqslant r$ and $p_{n_q} \leqslant q$. Let $m = \max\{n_r, n_q\}$. Then p_m is a common extension.

Clearly G is $\mathcal{D}$-generic. ☐

[Lecture 20, 2024-01-15]

Idea. *We want to add a new object that satisfies certain condition. The elements of the forcing notion correspond to approximations of this object.*

A filter picks some information which we want to be true. Being a filter ensures that this information does not contradict itself.

Definition 3.3. Assume that M is a transitive model of ZFC, and $\mathbb{P} \in M$ a poset. $G \subseteq \mathbb{P}$ is said to be **M-generic for $\mathbb{P}$** if whenever $D \subseteq \mathbb{P}$ is dense and in M, then $G \cap D \neq \emptyset$.

Remark 3.3.53. That is the same as being $\{D \subseteq \mathbb{P} \text{ dense } : D \in M\}$-generic with generic defined as in Definition 3.1.

Definition 3.4 (Cohen Forcing)**.** Let $\mathbb{P}$ be the set of finite partial function p from ω to 2, i.e. $\mathcal{P} = 2^{<\omega}$.

The order on $\mathbb{P}$ is described by $q \leqslant p :\iff q \supseteq p$. $\mathbb{P}$ is called the **Cohen forcing**.

Fact 3.4.54. Assume $X \subseteq 2^\omega$ is countable, Then there is $x \in 2^\omega \setminus X$.

Of course we already know that, but let's use it to test our machinery:

Proof. Assume that $X = \{x_n : n \in \omega\}$ is an enumeration of X. Let $D_n = \{p \in \mathbb{P} : \exists i \in \operatorname{dom}(\mathbb{P}).\ x_n(i) \neq p(i)\}$. This makes sure that we get a "new" element not belonging to X.

Claim 1. *D_n is dense in $\mathbb{P}$.*

Subproof. Assume $q \in \mathbb{P}$. Let $i = 1 + \max(\mathrm{dom}(q))$. Note that $i \notin \mathrm{dom}(q)$. Let $p = q \cup \{(i, 1 - x_n(i))\}$. Then $p \in D_n$. ■

Let $E_i = \{p \in \mathbb{P} : i \in \mathrm{dom}(\mathbb{P})\}$. This makes sure that our "new" element is defined everywhere.

Claim 2. *$\forall i < \omega.\ E_i \subseteq \mathbb{P}$ is dense.*

Subproof. Assume $q \in \mathbb{P}$. If $i \in \mathrm{dom}(q)$ pick $p = q \in E_i$. If $i \notin \mathrm{dom}(q)$, let $p = q \cup \{(i, 0)\} \in E_i$. ■

Let $\mathcal{D} = \{D_n : n < \omega\} \cup \{E_i : i < \omega\}$. This is a countable subset of dense sets. By Lemma 3.2 there is a $\mathcal{D}$-generic filter G. Let $y = \bigcup G$. Note that y is a function, since any two elements of G are compatible. □

Note that the "new" element did already exists, so we used forcing language to find it but didn't actually do anything.

Lemma 3.5. Let M be a transitive model of ZFC and let $\mathbb{P} = (P, \leqslant) \in M$. Let $D \subseteq \mathbb{P}$, $D \in M$, $p \in \mathbb{P}$. Then

(1) $\mathbb{P}$ is a partial order iff $M \models$ "$\mathbb{P}$ is a partial order".

(2) D is dense in $\mathbb{P}$ iff $M \models$ "D is dense in $\mathbb{P}$".

(3) D is dense below p iff $M \models$ "D is dense below p" (this only makes sense if $p \in M$).

Proof. All the definitions are Δ_0, so we can apply Lemma 2.114. □

Definition 3.6. Assume that M is a transitive model of ZFC and $\mathbb{P} \in M$ is a poset. $G \subseteq \mathbb{P}$ is called a **$\mathbb{P}$-generic filter over M** or **M-generic filter for $\mathbb{P}$** if

$$\forall D \in M.\ ((D \subseteq \mathbb{P} \text{ is dense}) \implies G \cap D \neq \varnothing).$$

Corollary 3.7. If M is a countable transitive model of ZFC, $\mathbb{P} \in M$ is a poset and $p \in \mathbb{P}$, then there is an M-generic filter $G \subseteq \mathbb{P}$ with $p \in G$.

Remark 3.7.55. The filter usually exists outside of M. M itself does not think that M is countable, since $M \models \mathsf{ZFC}$. But from the outside, we see that M is countable, so we can find a filter.

Definition 3.8. Assume that $\mathbb{P}$ is a poset. $\mathbb{P}$ is said to be **atomless** if for all $p \in \mathbb{P}$ there are $q, r \in \mathbb{P}$ such that

(1) $q \leqslant p$, $r \leqslant p$,

(2) $q \perp r$.

Example 3.9. The Cohen Forcing (3.4) is atomless.

Usually we are only interested in atomless partial orders.

Lemma 3.10. Assume that M is a transitive model of ZFC, $\mathbb{P} \in M$ an atomless poset and let $G \subseteq \mathbb{P}$ be M-generic for $\mathbb{P}$. Then $G \notin M$.

Proof. Towards a contradiction assume $G \in M$. Define $D := \mathbb{P} \setminus G$. We'll show that $D \subseteq \mathbb{P}$ is dense, which is a contradiction, since G was assumed to be M-generic. Let $q \in \mathbb{P}$ and let r, s be two extensions of q such that $r \perp s$. These exist because $\mathbb{P}$ is atomless. Since G is a filter, it can contain at most one of $\{r, s\}$, wlog. $s \notin G$. In particular, $s \in D$ and $s \leqslant q$. Hence D is dense in $\mathbb{P}$. □

Lemma 3.11. Assume that M is a transitive model of ZFC, $\mathbb{P} \in M$ a poset, $G \subseteq \mathbb{P}$ an M-generic filter and $p \in G$.

If D is dense below p, then $G \cap D \neq \varnothing$.

Proof. Let $E = D \cup \{q \in \mathbb{P} : q \perp p\}$. $E \subseteq \mathbb{P}$ is dense: Let $r \in \mathbb{P}$.

- If $r || p$ let $s \leqslant r, p$. Since D is dense below p, there exists $\overline{s} \in D$ such that $\overline{s} \leqslant s$. Since $D \subseteq E$, $\overline{s} \in E$.
- If $r \perp p$, then it is obvious that $r \in E$.

Since $E \in M$, $G \cap E \neq \varnothing$.

$$\begin{aligned} & G \cap (D \cup \{q \in \mathbb{P} : q \perp p\}) \neq \varnothing \\ \Longrightarrow \quad & (G \cap D) \cup \underbrace{(G \cap \{q \in \mathbb{P} : q \perp p\})}_{\varnothing} \neq \varnothing \end{aligned}$$

□

Definition 3.12. Assume that $\mathbb{P}$ is a poset.

(1) $A \subseteq \mathbb{P}$ is said to be an **antichain** iff for all $p \neq q$ in A, $p \perp q$.

(2) An antichain $A \subseteq \mathbb{P}$ is a **maximal antichain** iff $\forall r \in \mathbb{P}$, there exists $a \in \mathbb{P}$ such that $p || r$.

(3) $X \subseteq \mathbb{P}$ is said to be **open** if $\forall p \in X.\ \forall q \leqslant p.\ q \in X$.

Remark 3.12.56. Note that if A is a maximal antichain in $\mathbb{P}$, then it is maximal in $(\{A \subseteq \mathbb{P} : A \text{ is an antichain}\}, \subseteq)$. Using (C), every antichain can be extend to a maximal antichain.

The statement "A is an antichain" is Δ_0.

Note that "every antichain of $\mathbb{P}$ is countable" is not necessarily absolute between transitive models of ZFC.

Lemma 3.13. Assume that M is a transitive model of ZFC, $\mathbb{P} \in M$ a poset and $G \subseteq \mathbb{P}$ a filter. Then the following are equivalent:

(1) G is $\mathbb{P}$-generic over M.

(2) $G \cap A \neq \varnothing$ for every maximal antichain $A \in M$.

(3) $G \cap D \neq \varnothing$ for every dense open $D \in M$ with $D \subseteq \mathbb{P}$

We'll prove this next time.

[Lecture 21, 2024-01-18]

Goal. *We want to show that certain statements are consistent with* ZFC *(or* ZF*), for instance* CH.

We start with a model M of ZFC*. Usually we want M to be transitive.*

We want to enlarge M to get a bigger model, where our desired statement holds, i.e. add more reals to violate CH.

However we need to do this in a somewhat controlled way, so we can't just do it the way one builds field extensions. In particular, when trying to violate CH *we need to make sure that we don't collapse cardinals.*

Remark 3.13.57. The idea behind forcing is clever. Unfortunately an easy "how could I have come up with this myself"-approach does not seem to exist.

Remark 3.13.58. How can a countable transitive model M even exist?

M believes some statements that are wrong from the outside perspective. For example there exists $\aleph_1^M \in M$ such that $M \models x = \aleph_1$. $\aleph_1^M$ is indeed an ordinal (since being an ordinal is a Σ_0-statement). However $\aleph_1^M$ is countable, since M is countable and transitive. This is fine. (Note that "$\aleph_1^M$ is uncountable" is a Π_1-statement.)

Idea (The method of **forcing**)**.** *Start with M, a countable transitive model of* ZFC *and let $\mathbb{P} \in M$ be a partial order, where $p \leqslant q$ means that p has "more information" than q.*

A filter $g \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M iff $g \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$, $D \in M$.

Next steps:

(1) Define the ***forcing extension*** *$M[g]$.*

(2) Show that $M[g] \models \mathsf{ZFC}$.

(3) Determine other facts about (the theory of) $M[g]$. This depends on the partial order $\mathbb{P}$ we chose in the beginning (and maybe M).

Example 3.14 (Prototypical example)**.** Let $\mathbb{P} = 2^{<\omega}, p \leqslant q :\iff p \supseteq q$ be Cohen forcing, often denoted $\mathbb{C}$.

Let M be a countable transitive model of ZFC. Since the definition of $\mathbb{C}$ is simple enough, $\mathbb{C} \in M$. Let g be $\mathbb{C}$-generic over M.

Claim 1. *For each $n \in \omega$, the set $D_n := \{p \in \mathbb{C} : n \in \operatorname{dom}(p)\}$ is dense.*

Subproof. This is trivial. ■

Claim 2. *$D_n \in M$.*

Subproof. The definition of D_n is absolute. ■

Claim 3. *If $p, q \in g \cap D_n$, then $p(n) = q(n)$.*

Subproof. g is a filter, so p and q are compatible. $p, q \in D_n$ makes sure that $p(n)$ and $q(n)$ are defined. ■

Let $x = \bigcup g$. By Claim 3, $x \in 2^{\leqslant\omega}$. By Claim 1 and Claim 2, we have $g \cap D_n \neq \emptyset$ for all $n < \omega$, hence $n \in \operatorname{dom}(x)$ for all $n < \omega$. So $x \in 2^{\omega}$.

Claim 4. *Let $z \in 2^{\omega}$, $z \in M$. Then $D^z = \{p \in \mathbb{C} : \exists n \in \operatorname{dom}(p).\ p(n) \neq z(n)\}$ is dense.*

Subproof. Trivial. ■

Claim 5. *$D^z \in M$ for all $z \in 2^{<\omega}$ with $z \in M$. Therefore, $g \cap D^z \neq \emptyset$ for all $z \in M$, $z \colon 2^{<\omega}$. Hence $x \neq z$ for all $z \in M$, $z \in 2^{<\omega}$. In other words $x \notin M$.*

The new real x does not do too much damage to M when adding it.[a] (Some reals would completely kill the model.)

Now let α be an ordinal in M. Let

$$\begin{aligned}\mathbb{C}(\alpha) := \{p\colon & p \text{ is a function with domain } \alpha,\\ & p(\xi) \in \mathbb{C} \text{ for all } \xi < \alpha,\\ & \{\xi < \alpha : p(\xi) \neq \varnothing\} \text{ is finite}\}\end{aligned}$$

(α many copies of $\mathbb{C}$ with **finite support**).

For $p, q \in \mathbb{C}(\alpha)$ define $p \leqslant q :\iff \forall \xi < \alpha.\ p(\xi) \supseteq q(\xi)$. We have $\mathbb{C}(\alpha) \in M$

Let g be $\mathbb{C}(\alpha)$-generic over M. Let $x_\xi = \bigcup\{p(\xi) : p \in g\}$ for $\xi < \alpha$. $x_\xi \in 2^\omega$: For each $n < \omega$ and $\xi < \alpha$,

$$D_{n,\xi} := \{p \in \mathbb{C}(\alpha) : n \in \operatorname{dom}(p(\xi))\} \in M$$

and $D_{n,\xi}$ is dense.

Claim 6. *For all $\xi, \eta < \alpha$, $\xi \neq \eta$,*

$$D^{\xi,\eta} := \{p \in \mathbb{C}(\alpha) : \exists n \in \operatorname{dom}(p(\xi)) \cap \operatorname{dom}(p(\eta)).\ p(\xi)(n) \neq p(\eta)(n)\}$$

we have that $D^{\xi,\eta} \in M$ and is $D^{\xi,\eta}$ dense.

Therefore if $\xi \neq \eta$, $x_\xi \neq x_\eta$.

Currently this is not very exciting, since we only showed that for a countable transitive model M, there is a countable set of reals not contained in M. The interesting point will be, that we can actually add these reals to M.

[a] We still need to make this precise.

Next we want to define $M[g]$.

[Lecture 22, 2024-01-22]

Warning† 3.14.59. Forcing will not be relevant for the exam. Because of a lack of time, this is more of an outlook than a thorough presentation of the material.

For the rest of the section, let us fix a transitive model M of ZFC a partial order $\mathbb{P}$ and an M-generic filter g.

Definition 3.15 ($\mathbb{P}$-names)**.** For an ordinal $\alpha \in M$[a], let $M^{\mathbb{P}}_\alpha$, the **$\mathbb{P}$-names** in M of rank $\leqslant \alpha$, be defined as follows:

$$\tau \in M^{\mathbb{P}}_\alpha :\iff \tau \in M \wedge \tau \subseteq \mathbb{P} \times \bigcup\{M^{\mathbb{P}}_\beta : \beta < \alpha\},$$

i.e. the elements of $\tau \in M^{\mathbb{P}}_\alpha$ are of the form (p, σ), where $p \in \mathcal{P}$ and $\sigma \in M^{\mathbb{P}}_\beta$

for some $\beta < \alpha$.

Finally $M^{\mathbb{P}} = \bigcup\{M^{\mathbb{P}}_\alpha : \alpha \in M\}$.

[a]Recall that $\mathrm{Ord}_M = \mathrm{Ord} \cap M$.

Let R be the relation on $M^{\mathbb{P}}$ defined by $\sigma R \tau$ iff $\exists p \in \mathbb{P}.\ (p, \sigma) \in \tau$. If $\tau \in M^{\mathbb{P}}$ and $(p, \sigma) \in \tau$, then $\sigma \in \{p, \sigma\} \in (p, \sigma) \in \tau$, so the relation R is well founded.

Definition 3.16. Let $\tau \in M^{\mathbb{P}}_\alpha$. Then τ^g, the **g-interpretation of τ**, is defined to be

$$\{\sigma^g : \exists p \in g.\ (p, \sigma) \in \tau\}.$$

Definition 3.17. $M[g]$, the forcing extension of M given by g, is

$$\{\tau^g : \tau \in M^{\mathbb{P}}\}.$$

Lemma 3.18. $M[g]$ is transitive.

Proof. Trivial! □

Lemma 3.19. $M \cup \{g\} \subseteq M[g]$.

Proof. For all $x \in M$ we need to find a name $\check{x}$ such that $\check{x}^g = x$.

We can recursively (along $\in$) define

$$\check{x} = \{(p, \check{y}) : p \in \mathbb{P} \wedge y \in x\}.$$

By induction, $\check{x} \in M$ for all $x \in M$.

Claim 1. $\check{x}^g = x$.

Subproof. Recall that $\mathbb{P} \neq \varnothing$. Inductively, we get

$$\begin{aligned} \check{x}^g &= \{\check{y}^g : \exists p \in g.\ (p, \check{y}) \in \check{x}\} \\ &\overset{\text{induction}}{=} \{y : \exists p \in g.\ (p, \check{y}) \in \check{x}\} \\ &\overset{\text{definition of } \check{x}}{=} \{y : y \in x\} = x. \end{aligned}$$

■

So $M \subseteq M[g]$.

We also need a name for g. Let $\dot{g} := \{(p, \check{p}) : p \in \mathbb{P}\}$.

Indeed

$$\begin{aligned}\dot{g}^g &= \{\check{p}^g : \exists p \in g.\ (p, \check{p}) \in \dot{g}\}\\ &= \{p : p \in g\} = g.\end{aligned}$$

□

Lemma 3.20. $M[g] \models$ (Ext), (Fund), (Inf), (Pair), (Union).

Proof.

- (Ext):

  The formula $\forall x.\ \forall y.\ ((\forall z \in x.\ z \in y \land \forall z \in y.\ z \in x) \to x = y)$ is Π_1, hence it is true in $M[g]$ by Lemma 2.115.

- (Fund): Again,

  $$\forall x.\ (\exists y \in x.\ y = y \to \exists y \in x.\ \forall z \in y.\ z \notin x)$$

  is Π_1.

- (Inf) can be written as

  $$\exists x.\ (\underbrace{\neq \in x \land \forall y \in x. y \cup \{y\} \in x}_{\Sigma_0}).$$

  We have $\omega \in M \subseteq M[g]$, so $M[g] \models$ (Inf).

- (Pair): Let us assume $x, y \in M[g]$, say $x = \tau^g$ and $y = \sigma^g$. Let $\pi = \{(p, \tau) : p \in \mathbb{P}\} \cup \{(p, \sigma) : p \in \mathbb{P}\} \in M^{\mathbb{P}}$. Then $\pi^g = \{\tau^g, \sigma^g\} = \{x, y\}$, so $\{x, y\} \in M[g]$. As a $\mathcal{L}_\in$-statement, $z = \{x, y\}$ is Σ_0, so $M[g] \models$ "$\{x, y\}$ is the pair of x and y". Hence $M[g] \models$ (Pair).

- (Union): Similar to (Pair).

□

Still missing are

- (Pow),
- (Aus),
- (Rep),
- (C).

Definition 3.21 (**Forcing relation**)**.** Let M be a countable transitive model of ZFC and let $\mathbb{P} \in M$ be a partial order. Let $p \in \mathbb{P}$ and let φ be a $\mathcal{L}_\in$-formula.

Let $\tau_1, \ldots, \tau_k \in M^{\mathbb{P}}$ be names.

We say that p **forces** $\varphi(\tau_1, \dots, \tau_k)$,

$$p \Vdash^{\mathcal{P}}_M \varphi(\tau_1, \dots, \tau_k),$$

if for all $h \subseteq \mathbb{P}$ which are $\mathbb{P}$-generic over M with $p \in h$,

$$M[h] \models \varphi(\tau_1^h, \dots, \tau_k^h).$$

Theorem 3.22. Fix an $\mathcal{L}_\in$-formula φ. Then the relation

$$R = \{(p, \tau_1, \dots, \tau_k : p \Vdash^{\mathbb{P}}_M \varphi(\tau_1, \dots, \tau_k)\}$$

is definable over M (in the parameter $\mathbb{P}$).

Proof. Omitted. □

Theorem 3.23 (Forcing Theorem). Let M, $\mathbb{P}$, g, be as above, let φ be a formula, and let $\tau_1, \dots, \tau_k \in M^{\mathbb{P}}$. Then the following are equivalent:

(1) $M[g] \models \varphi(\tau_1^g, \dots, \tau_k^g)$.

(2) There is some $p \in g$ with

$$p \Vdash^{\mathbb{P}}_M \varphi(\tau_1, \dots, \tau_k).$$

Proof. Omitted. □

Theorem 3.24. $M[g] \models \mathsf{ZFC}$.

Proof. We have already shown a part of this in Lemma 3.20.

Let us show that $M[g] \models$ (Aus), the rest is similar and left as an exercise.[9]

Let φ be a formula, let $a, x_1, \dots, x_k \in M[g]$. We need to see

$$M[g] \models \exists y.\ y = \{z \in a : \varphi(z, x_1, \dots, x_k)\}.$$

If suffices to show that there is some $y \in M[g]$ with $y = \{z \in a : M[g] \models \varphi(z, x_1, \dots, x_k)\}$.

For this, let us construct a name for y. Let $a = \tau^g$, $x_i = \sigma_i^g$.

Let

$$\pi = \{(p, \rho) : \exists \overline{p} > p.\ (\overline{p}, \rho) \in \tau \wedge p \Vdash^{\mathbb{P}}_M \varphi(\rho, \sigma_1, \dots, \sigma_k)\}.$$

We have $\pi \in M$, since the relation $\Vdash^{\mathbb{P}}_M$ can be defined in M.

[9] or done next semester in Logic IV!

Let $z \in a$ such that $M[g] \models \varphi(z, x_1, \ldots, x_n)$. We have $z = \rho^g$ for some ρ and there is $\overline{p} \in g$ with $(\overline{p}, \rho) \in \pi$. Now $M[g] = \varphi(\rho^g, \sigma_1^g, \ldots \sigma_k^g)$.

Let $p' \Vdash_M^{\mathbb{P}} \varphi(\rho, \sigma_1, \ldots, \sigma_k)$, where $p' \in g$. We have $p', \overline{p} \in g$, so there is some $p \leqslant p', \overline{p}$ with $p \in g$. Then $(p, \rho) \in \pi$, so $\rho^g \in \pi^g$.

This shows that

$$\{z \in a : M[g] \models \varphi(z, x_1, \ldots, x_k)\} \subseteq \pi^g.$$

The other inclusion is easy.

$\square$

[Lecture 23, 2024-01-25]

Goal. *We want to construct a model of* ZFC *such that* $2^{\aleph_0} \geqslant \aleph_2$.

Let M be a countable transitive model of ZFC. Suppose that $M \models \mathrm{CH}$ (otherwise we are done).

Let $\alpha = \omega_2^M$.

Let $\mathbb{C}(\alpha) := \{p : p\colon \alpha \to \mathbb{C}$ is a function such that $\{\xi < \alpha : p(\xi) \neq \varnothing\}$ is finite$\}$, ordered by $p \leqslant_{\mathbb{C}(\alpha)} q$ iff $p(\xi) \leqslant_{\mathbb{C}} q(\xi)$ for all $\xi < \alpha$.

Recall that $\mathbb{C}$ is the set of finite sequences of natural numbers ordered by $p \leqslant_{\mathbb{C}} q$ iff $p \supseteq q$.

Let g be $\mathbb{C}(\alpha)$-generic over M. For $\xi < \alpha$ let $x_\xi = \bigcup\{p(\xi) : p \in g\}$. We have already seen that $x_\xi\colon \omega \to \omega$ is a function and $x_\xi \neq x_\eta$ for $\xi \neq \eta$.

We have $M[g] \models$ ZFC.[10] As $g \in M[g]$, we have $\langle x_\xi : \xi < \alpha\rangle \in M[g]$. Therefore $M[g] \models$ "$2^{\aleph_0} \geqslant \alpha$". Also $\alpha = \omega_2^M$. However the proof is not finished yet, since we need to make sure, that $M[g]$ does not collapse cardinals.

We only have $M[g] \models 2^{\aleph_0} \geqslant \aleph_2^M$, i.e. we need to see $\aleph_2^{M[g]} = \aleph_2^M$.

Claim 7. *Every cardinal of M is still a cardinal of $M[g]$.*

This suffices, because then $\aleph_0^M = \aleph_0^{M[g]}$, $\aleph_1^M = \aleph_1^{M[g]}$, $\aleph_2^M = \aleph_2^{M[g]}$, ...

Definition 3.25. Let $(\mathbb{P}, \leqslant)$ be a partial order. We say that $\mathbb{P}$ has the **countable chain condition (c.c.c.)**[a] iff there is no uncountable antichain, i.e. every uncountable $V \subseteq \mathbb{P}$ contains compatible $p \neq q$.

[a] it should really be the "countable antichain condition"

We shall prove:

Claim 8. *For all β, $\mathbb{C}(\beta)$ has the c.c.c.*

[10] We only handwaved this step.

Claim 9. *If $\mathbb{P} \in M$ and $M \models$ " $\mathbb{P}$ has the c.c.c." and h is generic over M, then all M-cardinals are still $M[h]$ cardinals.* [11]

Proof of Claim 9. Suppose not. Let κ be minimal such that $M \models$ "κ is a cardinal", but $M[h] \models$ "κ is not a cardinal". Then $\kappa = (\lambda^+)^M$ for some unique M-cardinal $\lambda < \kappa$. By minimality, λ is also an $M[h]$-cardinal.

Let $f \in M[h]$ be such that $M[h] \models$ "f is a surjection from λ onto κ". There is a name $\tau \in M^{\mathbb{P}}$ with $\tau^h = f$.

We then have some $p \in h$ with $p \Vdash_M^{\mathbb{P}}$ "τ is a surjection from $\check{\lambda}$ onto $\check{\kappa}$".

Let $\xi < \lambda$. Consider $X_\xi := \{\eta < \kappa : \exists q \leqslant_{\mathbb{P}} p.\ q \Vdash \tau(\check{\xi}) = \check{\eta}\} \in M$.

X_ξ is countable in M by the following argument (in M): For every $\eta \in X_\xi$, let $q_\eta \leqslant p$ be such that $q_\xi \Vdash_M^{\mathbb{P}} \tau(\check{\xi}) = \check{\eta}$. The set $\{q_\eta : \eta \in X_\xi\}$ is an antichain as for $\eta_1 \neq \eta_2$ we have that $q_{\eta_i} \Vdash \tau(\check{\xi}) = \check{\eta}_i$, so they are not compatible. So $\{q_\eta : \eta \in X_\xi\}$ is countable by the c.c.c. Thus X_ξ is countable.

Therefore we may define a function in M

$$F\colon \lambda \times \omega \longrightarrow \kappa$$

such that for all $\xi < \lambda$

$$\{F(\xi, n) : n < \omega\} = X_\xi.$$

F is surjective since f is surjective: For $\eta < \kappa$, there is some $\xi < \lambda$ such that $M[h] \models$ "$f(\xi) = \eta$", there is some $\overline{q} \in h$ with $\overline{q} \Vdash_M^{\mathbb{P}} \tau(\check{\xi}) = \check{\eta}$. Pick $q \leqslant \overline{q}, p$. This shows $\eta \in X_\xi$ hence $\eta = F(\xi, n)$ for some n. But $|\lambda \times \omega| = |\lambda| = \lambda$, so in M there is a surjection $F'\colon \lambda \to \kappa$, but κ is a cardinal in M $\lightning$. $\square$

Proof of Claim 8. Omitted. $\square$

[11] Being a cardinal is Π_1, so $M[h]$ cardinals are always M cardinals.

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