Logic 3: Abstract Topological Dynamics and Descriptive Set Theory

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Contents

These are my notes on the lecture Logic 3: Abstract Topological Dynamics and Descriptive Set Theory taught by ALEKSANDRA KWIATKOWSKA in the summer term 2023 at the University Münster.

> **Warning 0.1.** This is not an official script. The official lecture notes can be found here.

If you find errors or want to improve something, please send me a message: `lecturenotes@jrpie.de`.

Many thanks to MIRKO BARTSCH for providing notes on lectures I could not attend!

This notes follow the way the material was presented in the lecture rather closely. Additions (e.g. from exercise sheets) and slight modifications have been marked with †.

[Lecture 01, 2023-10-10]

1 Introduction

Definition 1.1. Let X be a nonempty topological space. We say that X is a **Polish space** if X is

- **separable**, i.e. there exists a countable dense subset, and
- **completely metrisable**, i.e. there exists a complete metric on X which induces the topology.

Note that Polishness is preserved under homeomorphisms, i.e. it is really a topological property.

Example 1.2.

- $\mathbb{R}$ is a Polish space,
- $\mathbb{R}^n$ for finite n is Polish,
- $[0, 1]$,
- any countable discrete topological space,
- the completion of any separable metric space considered as a topological space.

Polish spaces behave very nicely. We will see that uncountable polish spaces have size $2^{\aleph_0}$. There are good notions of big (comeager) and small (meager).

1.1 Topology background

Recall the following notions:

Definition 1.3 (**product topology**)**.** Let $(X_i)_{i\in I}$ be a family of topological spaces. Consider the set $\prod_{i\in I} X_i$ and the topology induced by basic open sets $\prod_{i\in I} U_i$ with $U_i \subseteq X_i$ open and $U_i \subsetneq X_i$ for only finitely many i.

Fact 1.3.1. Countable products of separable spaces are separable.

Definition 1.4. A topological space X is **second countable**, if it has a countable base.

Let X be a topological space. If X is second countable, it is also separable. However the converse of this does not hold.

Example 1.5. Let X be an uncountable set. Take $x_0 \in X$ and consider the topology given by

$$\tau = \{U \subseteq X | U \ni x_0\} \cup \{\varnothing\}.$$

Then $\{x_0\}$ is dense in X, but X is not second countable.

Example 1.6 (Sorgenfrey line)**.** Consider $\mathbb{R}$ with the topology given by the basis $\{[a, b) : a, b \in \mathbb{R}\}$. This is T3, but not second countable and not metrizable.

Fact 1.6.2. For metric spaces, the following are equivalent:

- separable,
- second-countable,
- **Lindelöf** (every open cover has a countable subcover).

Fact 1.6.3. Compact Hausdorff spaces are **normal** (T4) i.e. two disjoint closed subsets can be separated by open sets.

Fact 1.6.4. For a metric space, the following are equivalent:

- compact,
- **sequentially compact** (every sequence has a convergent subsequence),
- complete and **totally bounded** (for all $\varepsilon > 0$ we can cover the space with finitely many ε-balls).

Theorem 1.7 (Urysohn's metrisation theorem)**.** Let X be a topological space. If X is

- second countable,
- Hausdorff and
- regular (T3)

then X is metrisable.

Fact 1.7.5. If X is a compact Hausdorff space, the following are equivalent:

- X is Polish,
- X is metrisable,
- X is second countable.

1.2 Some facts about polish spaces

Fact 1.7.6. Let (X, τ) be a topological space. Let d be a metric on X. We will denote the topology induces by this metric as τ_d. To show that $\tau = \tau_d$, it is equivalent to show that

- every open d-ball is in τ ($\implies \tau_d \subseteq d$) and
- every open set in τ is a union of open d-balls.

To show that $\tau_d = \tau_{d'}$ for two metrics d, d', suffices to show that open balls in one metric are unions of open balls in the other.

Notation 1.7.7. We sometimes[a] denote $\min(a, b)$ by $a \wedge b$.

[a] only in this subsection?

Proposition 1.8. Let (X, τ) be a topological space, d a metric on X compatible with τ (i.e. it induces τ).

Then $d' := \min(d, 1)$ is also a metric compatible with τ.

Proof. To check the triangle inequality:

$$\begin{aligned} d(x, y) \wedge 1 &\leqslant (d(x, z) + d(y, z)) \wedge 1 \\ &\leqslant (d(x, z) \wedge 1) + (d(y, z) \wedge 1) . \end{aligned}$$

For $\varepsilon \leqslant 1$ we have $B'_\varepsilon(x) = B_\varepsilon(x)$ and for $\varepsilon > 1$, $B'_\varepsilon(x) = X$.

Since d is complete, we have that d' is complete. $\square$

Proposition 1.9. Let A be a Polish space. Then A^ω Polish.

Proof. Let A be separable. Then A^ω is separable. (Consider the basic open sets of the product topology).

Let $d \leqslant 1$ be a complete metric on A. Define D on A^ω by

$$D((x_n), (y_n)) := \sum_{n<\omega} 2^{-(n+1)} d(x_n, y_n).$$

Clearly $D \leqslant 1$. It is also clear, that D is a metric.

We need to check that D is complete: Let $(x_n)^{(k)}$ be a Cauchy sequence in A^ω. Consider the pointwise limit (a_n). This exists since $x_n^{(k)}$ is Cauchy for every fixed n. Then $(x_n)^{(k)} \xrightarrow{k\to\infty} (a_n)$. □

Definition 1.10 (Our favourite Polish spaces)**.**

- $2^{\mathbb{N}}$ is called the **Cantor set**. (Consider 2 with the discrete topology)
- $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ is called the **Baire space**. ($\mathbb{N}$ with descrete topology)
- $\mathbb{H} := [0,1]^{\mathbb{N}}$ is called the **Hilbert cube**. ($[0,1] \subseteq \mathbb{R}$ with the usual topology)

Proposition 1.11. Let X be a separable, metrisable topological space[a]. Then X topologically embeds into the **Hilbert cube**, i.e. there is an injective $f : X \hookrightarrow [0,1]^\omega$ such that $f : X \to f(X)$ is a homeomorphism.

[a] e.g. Polish, but we don't need completeness.

Proof. X is separable, so it has some countable dense subset, which we order as a sequence $(x_n)_{n\in\omega}$.

Let d be a metric on X which is compatible with the topology. W.l.o.g. $d \leqslant 1$ (by Proposition 1.8). Define

$$\begin{aligned} f\colon X &\longrightarrow [0,1]^\omega \\ x &\longmapsto (d(x,x_n))_{n<\omega} \end{aligned}$$

Claim 1. *f is injective.*

Subproof. Suppose that $f(x) = f(y)$. Then $d(x,x_n) = d(y,y_n)$ for all n. Hence $d(x,y) \leqslant d(x,x_n) + d(y,x_n) = 2d(x,x_n)$. Since (x_n) is dense, we get $d(x,y) = 0$. ■

Claim 2. *f is continuous.*

Subproof. Consider a basic open set in $[0,1]^\omega$, i.e. specify open sets $U_1,\ldots,U_n$ on finitely many coordinates. $f^{-1}(U_1 \times \ldots \times U_n \times \ldots)$ is a finite intersection of open sets, hence it is open. ■

Claim 3. *f^{-1} is continuous.*

Subproof. Consider $B_\varepsilon(x_n) \subseteq X$ for some $n \in \mathbb{N}$, $\varepsilon > 0$. Then $f(U) = f(X) \cap [0,1]^n \times [0,\varepsilon) \times [0,1]^\omega$ is open[1]. ■

□

Proposition 1.12. Countable disjoint unions of Polish spaces are Polish.

Proof. Define a metric in the obvious way. □

Proposition 1.13. Closed subspaces of Polish spaces are Polish.

Proof. Let X be Polish and $V \subseteq X$ closed. Let d be a complete metric on X. Then $d|_V$ is complete. Subspaces of second countable spaces are second countable. □

Definition 1.14. Let X be a topological space. A subspace $A \subseteq X$ is called G_δ[a], if it is a countable intersection of open sets.

[a]Gebietdurchschnitt

Next time: Closed sets are G_δ. A subspace of a Polish space is Polish iff it is G_δ

[Lecture 02, 2023-10-13]

Theorem 1.15. A subspace of a Polish space is Polish iff it is G_δ.

Remark 1.15.8. Closed subsets of a metric space (X,d) are G_δ.

Proof. Let $C \subseteq X$ be closed. Let $U_{\frac{1}{n}} := \{x | d(x,C) < \frac{1}{n}\}$. Clearly $C \subseteq \bigcap U_{\frac{1}{n}}$. Let $x \in \bigcap U_{\frac{1}{n}}$. Then $\forall n.\ \exists x_n \in C.\ d(x,x_n) < \frac{1}{n}$. The x_n converge to x and since C is closed, we get $x \in C$. Hence $C = \bigcap U_{\frac{1}{n}}$ is G_δ. □

Example 1.16. Let X be Polish. Let d be a complete metric on X.

a) If $Y \subseteq X$ is closed, then $(Y, d|_Y)$ is complete.

b) $Y = (0,1) \subseteq \mathbb{R}$ with the usual metric $d(x,y) = |x-y|$. Then $x_n \to 0$ is Cauchy in $((0,1),d)$.

   But
   $$d_1(x,y) := |x-y| + \left|\frac{1}{\min(x,1-x)} - \frac{1}{\min(y,1-y)}\right|$$

[1]as a subset of $f(X)$!

also is a complete metric on $(0,1)$ which is compatible with d.

We want to generalize this idea.

Proof of Theorem 1.15.

Claim 1.15.1. *If $Y \subseteq (X,d)$ is G_δ, then there exists a complete metric on Y.*

Proof of Claim 1.15.1. Let $Y = U$ be open in X. Consider the map

$$\begin{aligned} f_U \colon U &\longrightarrow \underbrace{X}_{d} \times \underbrace{\mathbb{R}}_{|\cdot|} \\ x &\longmapsto \left(x, \frac{1}{d(x,U^c)}\right). \end{aligned}$$

Note that $X \times \mathbb{R}$ with the

$$d_1((x_1,y_1),(x_2,y_2)) := d(x_1,x_2) + |y_1 - y_2|$$

metric is complete.

f_U is an embedding of U into $X \times \mathbb{R}$:

- It is injective because of the first coordinate.
- It is continuous since $d(x,U^c)$ is continuous and only takes strictly positive values.
- The inverse is continuous because projections are continuous.

So we have shown that U and the graph of $\tilde{f_U} \colon x \mapsto \frac{1}{d(x,U^c)}$ are homeomorphic. The graph is closed in $U \times \mathbb{R}$, because $\tilde{f_U}$ is continuous. It is closed in $X \times \mathbb{R}$ because $\tilde{f_U} \to \infty$ for $d(x,U^c) \to 0$.

Therefore we identified U with a closed subspace of the Polish space $(X \times \mathbb{R}, d_1)$.

$\square$

Let $Y = \bigcap_{n\in\mathbb{N}} U_n$ be G_δ. Consider

$$\begin{aligned} f_Y \colon Y &\longrightarrow X \times \mathbb{R}^{\mathbb{N}} \\ x &\longmapsto \left(x, \left(\frac{1}{\delta(x,U_n^c)}\right)_{n\in\mathbb{N}}\right) \end{aligned}$$

As for an open U, f_Y is an embedding. Since $X \times \mathbb{R}^{\mathbb{N}}$ is completely metrizable, so is the closed set $f_Y(Y) \subseteq X \times \mathbb{R}^{\mathbb{N}}$.

Claim 1.15.2. *If $Y \subseteq (X,d)$ is completely metrizable, then Y is a G_δ subspace.*

Proof of Claim 1.15.2. There exists a complete metric d_Y on Y. For every n, let $V_n \subseteq X$ be the union of all open sets $U \subseteq X$ such that

(i) $U \cap Y \neq \varnothing$,

(ii) $\operatorname{diam}_d(U) \leqslant \frac{1}{n}$,[2]

(iii) $\operatorname{diam}_{d_Y}(U \cap Y) \leqslant \frac{1}{n}$.

We want to show that $Y = \bigcap_{n \in \mathbb{N}} V_n$. For $x \in Y$, $n \in \mathbb{N}$ we have $x \in V_n$, as we can choose two neighbourhoods U_1 (open in Y) and U_2 (open in X) of x, such that $\operatorname{diam}_{d_Y}(U) < \frac{1}{n}$ and $U_2 \cap Y = U_1$. Additionally choose $x \in U_3$ open in X with $\operatorname{diam}_d(U_3) < \frac{1}{n}$. Then consider $U_2 \cap U_3 \subseteq V_n$. Hence $Y \subseteq \bigcap_{n \in \mathbb{N}} V_n$.

Now let $x \in \bigcap_{n \in \mathbb{N}} V_n$. For each n pick $x \in U_n \subseteq X$ open satisfying (i), (ii), (iii). From (i) and (ii) it follows that $x \in \overline{Y}$, since we can consider a sequence of points $y_n \in U_n \cap Y$ and get $y_n \xrightarrow{d} x$. For all n we have that $U_n' := U_1 \cap \ldots \cap U_n$ is an open set containing x, hence $U_n' \cap Y \neq \varnothing$. Thus we may assume that the U_i form a decreasing sequence. We have that $\operatorname{diam}_{d_Y}(U_n \cap Y) \leqslant \frac{1}{n}$. If follows that the y_n form a Cauchy sequence with respect to d_Y, since $\operatorname{diam}(U_n \cap Y) \xrightarrow{d_Y} 0$ and thus $\operatorname{diam}(\overline{U_n \cap Y}) \xrightarrow{d_Y} 0$. The sequence y_n converges to the unique point in $\bigcap_n \overline{U_n \cap Y}$. Since the topologies agree, this point is x. □

□

[Lecture 03, 2023-10-17]

1.3 Trees

Notation 1.16.9. Let $A \neq \varnothing$, $n \in \mathbb{N}$. Then

$$A^n := \{s \colon \{0, 1, \ldots, n-1\} \to A\}$$

is the set of n-element **sequences**. We often write $(s_0, s_1, \ldots, s_{n-1})$.

If $s = (s_0, \ldots, s_{n-1})$, then n is the **length** of s, denoted by $|s|$.

If $n = 0$ there exists only the empty sequence, i.e. $A^0 = \{\varnothing\}$ and $|\varnothing| = 0$.

We set

$$A^{<\mathbb{N}} := \bigcup_{n=0}^{\infty} A^n$$

and

$$A^{\mathbb{N}} := \{x \colon \mathbb{N} \to A\}.$$

If $s \in A^n$ and $m \leqslant n$, we let $s|_m := (s_0, \ldots, s_{m-1})$.

[2] The proof gets a little easier if we bound by $\frac{1}{2^n}$ instead of $\frac{1}{n}$, as that allows to simply take $U_n' := \bigcup_{m>n} U_m$, but both bounds work.

Let $s, t \in A^{<\mathbb{N}}$. We say that s is an **initial segment** of t (or t is an **extension** of s) if there exists an n such that $s = t|_{|s|}$. We write this as $s \subseteq t$.

We say that s and t are **compatible** if $s \subseteq t$ or $t \subseteq s$. Otherwise the are **incompatible**, we denote that as $s \perp t$.

The **concatenation** of $s = (s_0, \ldots, s_{n-1})$ and $t = (t_0, \ldots, t_{m-1})$ is the sequence $s ^\frown t := (s_0, \ldots, s_{n-1}, t_0, \ldots, t_{n-1})$

In the case of $t = (a)$ we also write $s ^\frown a$ for $s^\frown(a)$.

Similarly, if $x \in A^{\mathbb{N}}$ we can write $x = (x_0, x_1, \ldots)$. If $n \in \mathbb{N}$, $x|_n := (x_0, \ldots, x_{n-1})$, define extension, initial segments and concatenation of a finite sequence with an infinite one.

Definition 1.17. A **tree** on a set A is a subset $T \subseteq A^{<\mathbb{N}}$ closed under initial segments, i.e. if $t \in T, s \subseteq t \implies s \in T$. Elements of trees are called **nodes**.

A **leave** is an element of T, that has no extension in t.

An **infinite branch** of a tree T is $x \in A^{\mathbb{N}}$ such that $\forall n.\ x|_n \in T$.

The **body** of T is the set of all infinite branches:

$$[T] := \{x \in A^{\mathbb{N}} : \forall n. x|_n \in T\}.$$

We say that T is **pruned**, iff

$$\forall t \in T. \exists s \supsetneq t.\ s \in T.$$

Definition 1.18. A **Cantor scheme** on a set X is a family $(A_s)_{s \in 2^{<\mathbb{N}}}$ of subsets of X such that

i) $\forall s \in 2^{<\mathbb{N}}.\ A_{s^\frown 0} \cap A_{s^\frown 1} = \varnothing$.

ii) $\forall s \in 2^{<\mathbb{N}}, i \in 2.\ A_{s^\frown i} \subseteq A_s$.

Definition 1.19. A topological space is **perfect** if it has no isolated points, i.e. for any $U \neq \varnothing$ open, there $x \neq y$ such that $x, y \in U$.

Theorem 1.20. Let $X \neq \varnothing$ be a perfect Polish space. Then there is an embedding of the Cantor space $2^{\mathbb{N}}$ into X.

Proof. We will define a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ such that $\forall s \in 2^{<\mathbb{N}}$.

(i) $U_s \neq \varnothing$ and open,

(ii) $\operatorname{diam}(U_s) \leqslant 2^{-|s|}$,

(iii) $\overline{U_{s^\frown i}} \subseteq U_s$ for $i \in 2$.

We define U_s inductively on the length of s.

For $U_\varnothing$ take any non-empty open set with small enough diameter.

Given U_s, pick $x \neq y \in U_s$ and let $U_{s^\frown 0} \ni x$, $U_{s^\frown 1} \ni y$ be disjoint, open, of diameter $\leqslant \frac{1}{2^{|s|+1}}$ and such that $\overline{U_{s^\frown 0}}, \overline{U_{S^\frown 1}} \subseteq U_s$.

Let $x \in 2^{\mathbb{N}}$. Then let $f(x)$ be the unique point in X such that

$$\{f(x)\} = \bigcap_n U_{x|_n} = \bigcap_n \overline{U_{x|_n}}.$$

(This is nonempty as X is a completely metrizable space.) It is clear that f is injective and continuous. $2^{\mathbb{N}}$ is compact, hence f^{-1} is also continuous. □

Corollary 1.21. Every nonempty perfect Polish space X has cardinality $\mathfrak{c} = 2^{\aleph_0}$

Proof. Since the cantor space embeds into X, we get the lower bound. Since X is second countable and Hausdorff, we get the upper bound: Let $\langle U_n : n < \omega\rangle$ be a countable basis. Consider the injective function

$$\begin{aligned} f\colon X &\longrightarrow 2^\omega \\ x &\longmapsto \{n : x \in U_n\}. \end{aligned}$$

□

Theorem 1.22. Any Polish space is countable or it has cardinality $\mathfrak{c}$.

Proof. See Corollary A.7. □

Definition 1.23. A **Lusin scheme** on a set X is a family $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of X such that

(i) $A_{s^\frown i} \cap A_{s^\frown j} = \varnothing$ for all $j \neq i \in \mathbb{N}$, $s \in \mathbb{N}^{<\mathbb{N}}$.

(ii) $A_{s^\frown i} \subseteq A_s$ for all $i \in \mathbb{N}, s \in \mathbb{N}^{<\mathbb{N}}$.

Theorem 1.24. Let $X \neq \varnothing$ be a Polish space. Then there is a closed subset

$$D \subseteq \mathbb{N}^{\mathbb{N}} =: \mathcal{N}$$

and a continuous bijection $f\colon D \to X$ (the inverse does not need to be continuous).

Moreover there is a continuous surjection $g : \mathcal{N} \to X$ extending f.

Definition 1.25. An F_σ set is the countable union of closed sets, i.e. the complement of a G_δ set.

Observe.

- *Any open set is $F\sigma$.*
- *In metric spaces the intersection of an open and closed set is F_σ.*

Proof of Theorem 1.24. Let d be a complete metric on X. W.l.o.g. $\operatorname{diam}(X) \leqslant 1$. We construct a Lusin scheme $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ such that $F_s \subseteq X$ and

(i) $F_\varnothing = X$,

(ii) F_s is F_σ for all s.

(iii) The $F_{s^\frown i}$ partition F_s, i.e. $F_s = \bigsqcup_i F_{s^\frown i}$.

Furthermore we want that $\overline{F_{s^\frown i}} \subseteq F_s$ for all i.

(iv) $\operatorname{diam}(F_s) \leqslant 2^{-|s|}$.

Suppose we already have $F_s =: F$. We need to construct a partition $(F_i)_{i\in\mathbb{N}}$ of F with $\overline{F_i} \subseteq F$ and $\operatorname{diam}(F_i) < \varepsilon$ for $\varepsilon = 2^{-|s|-1}$, such that the F_i are F_σ.

Step 1 Write $F := \bigcup_{i\in\mathbb{N}} C_i$ for some closed sets C_i. W.l.o.g. $C_i \subseteq C_{i+1}$.

Let $F_i^0 := C_{i+1} \setminus C_i$. These F_i^0 are F_σ, and form a partition of F. Furthermore $\overline{F_i^0} \subseteq F$.

However the diameter might be too large. Fix $i \in \mathbb{N}$ and consider F_i^0. Cover it with countably many open balls $B_1, B_2, \ldots$ of diameter smaller than ε. The sets $D_i := F_i^0 \cap B_i \setminus (B_1 \cup \ldots \cup B_{i-1})$ are F_σ, disjoint and $F_i^0 = \bigcup_j D_j$.

[Lecture 04, 2023-10-20]

Remark 1.25.10. Some of the F_s might be empty.

Continuation of proof of Theorem 1.24. Take

$$D = \{x \in \mathcal{N} : \bigcap_n F_{x|_n} \neq \varnothing\}.$$

Since $\ldots \supseteq F_{x|_n} \supseteq \overline{F_{x|_{n+1}}} \supseteq F_{x|_{n+1}} \supseteq \ldots$ we have

$$\bigcap_n F_{x|_n} = \bigcap_n \overline{F_{x|_n}}.$$

$f: D \to X$ is determined by

$$\{f(x)\} = \bigcap_n F_{x|_n}$$

f is injective and continuous. The proof of this is exactly the same as in Theorem 1.20.

Claim 1.24.1. *D is closed.*

Proof of Claim 1.24.1. Let x_n be a series in D converging to x in $\mathcal{N}$.

Claim 1.24.1.1. *$(f(x_n))$ is Cauchy.*

Subproof. Let $\varepsilon > 0$. Take N such that $\operatorname{diam}(F_{x|_n}) < \varepsilon$. Take M such that for all $m \geqslant M$, $x_m|_N = x|_N$. Then for all $m, n \geqslant M$, we have that $f(x_m), f(x_n) \in F_{x|_N}$. So $d(f(x_m), f(x_n)) < \varepsilon$, i.e. $(f(x_n))$ is Cauchy.

■

Since (X, d) is complete, there exists $y = \lim_n f(x_n)$. Since for all $m \geqslant M$, $f(x_m) \in F_{x|_N}$, we get that $y \in \overline{F_{x|_N}}$.

Note that for $N' > N$ by the same argument we get $y \in \overline{F_{x|_{N'}}}$. Hence

$$y \in \bigcap_n \overline{F_{x|_n}} = \bigcap_n F_{x|_n},$$

i.e. $y \in D$ and $y = f(x)$. □

We extend f to $g: \mathcal{N} \to X$ in the following way:

Take $S := \{s \in \mathbb{N}^{<\mathbb{N}} : \exists x \in D, n \in \mathbb{N}.\ x|_n = s\}$. Clearly S is a pruned tree. Moreover, since D is closed, we have that[3]

$$D = [S] = \{x \in \mathbb{N}^{\mathbb{N}} : \forall n \in \mathbb{N}.\ x|_n \in S\}.$$

We construct a **retraction** $r: \mathcal{N} \to D$ (i.e. $r = \mathrm{id}$ on D and r is a continuous surjection). Then $g := f \circ r$.

To construct r, we will define $\varphi: \mathbb{N}^{<\mathbb{N}} \to S$ by induction on the length such that

- $s \subseteq t \implies \varphi(s) \subseteq \varphi(t)$,
- $|s| = \varphi(|s|)$,
- if $s \in S$, then $\varphi(s) = s$.

[3] cf. Sheet 3, Exercise 1 (A.3.1)

Let $\varphi(\varnothing) = \varnothing$. Suppose that $\varphi(t)$ is defined. If $t \frown a \in S$, then set $\varphi(t \frown a) := t \frown a$. Otherwise take some b such that $t \frown b \in S$ and define $\varphi(t \frown a) := \varphi(t) \frown b$.This is possible since S is pruned.

Let $r \colon \mathcal{N} = [\mathbb{N}^{<\mathbb{N}}] \to [S] = D$ be the function defined by $r(x) = \bigcup_n f(x|_n)$.

r is continuous, since $d_{\mathcal{N}}(r(x), r(y)) \leqslant d_{\mathcal{N}}(x, y)$. It is immediate that r is a retraction. □

2 Meager and Comeager Sets

Definition 2.1. Let X be a topological space, $A \subseteq X$. We say that A is **nowhere dense** (**nwd**), if $\operatorname{int}(\overline{A}) = \varnothing$. Equivalently

- $\overline{A}$ is nwd,
- $X \setminus \overline{A}$ is dense in X,
- $\forall \varnothing \neq U \overset{\text{open}}{\subseteq} X.\ \exists \varnothing \neq V \overset{\text{open}}{\subseteq} U.\ V \cap A = \varnothing$. (If we intersect A with an open U, then $A \cap U$ is not dense in U).

A set $B \subseteq X$ is **meager** (or **first category**), iff it is a countable union of nwd sets.

The complement of a meager set is called **comeager**.

Example 2.2. $\mathbb{Q} \subseteq \mathbb{R}$ is meager.

Notation 2.2.11. Let $A, B \subseteq X$. We write $A =^* B$ iff the **symmetric difference**, $A \mathbin{\triangle} B := (A \setminus B) \cup (B \setminus A)$, is meager.

Remark 2.2.12. $=^*$ is an equivalence relation.

Definition 2.3. A set $A \subseteq X$ has the **Baire property** (**BP**) if $A =^* U$ for some $U \overset{\text{open}}{\subseteq} X$.

Note that open sets and meager sets have the Baire property.

Example 2.4.

- $\mathbb{Q} \subseteq \mathbb{R}$ is F_σ.
- $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$ is G_δ.
- $\mathbb{Q} \subseteq \mathbb{R}$ is not G_δ: It is dense and meager, hence it can not be G_δ by the Baire Category Theorem (2.7).

[Lecture 05, 2023-10-31]

Fact 2.4.13.
- A set A is nwd iff $\overline{A}$ is nwd.
- If F is closed then F is nwd iff $X \setminus F$ is open and dense.
- Any meager set B is contained in a meager F_σ-set.

Proof.
- This follows from the definition as $\overline{\overline{A}} = \overline{A}$.
- Trivial.
- Let $B = \bigcup_{n<\omega} B_n$ be a union of nwd sets. Then $B \subseteq \bigcup_{n<\omega} \overline{B_n}$.

□

Definition 2.5. A ***σ*-algebra** on a set X is a collection of subsets of X such that:

- $\varnothing, X \in \mathcal{A}$,
- $A \in \mathcal{A} \implies X \setminus A \in \mathcal{A}$,
- $(A_i)_{i<\omega}, A_i \in \mathcal{A} \implies \bigcup_{i<\omega} A_i \in \mathcal{A}$.

Fact 2.5.14. Since $\bigcap_{i<\omega} A_i = (\bigcup_{i<\omega} A_i^c)^c$ we have that σ-algebras are closed under countable intersections.

Theorem 2.6. Let X be a topological space. Then the collection of sets with the Baire property is a σ-algebra on X.

It is the smallest σ-algebra containing all meager and open sets.

Proof of Theorem 2.6. Let $\mathcal{A}$ be the collection of sets with the Baire property. Since open sets have the Baire property, we have $\varnothing, X \in \mathcal{A}$.

Let $A_n \in \mathcal{A}$ for all $n < \omega$. Take U_n such that $A_n \triangle U_n$ is meager. Then

$$\left(\bigcup_{n<\omega} A_n\right) \triangle \left(\bigcup_{n<\omega} U_n\right)$$

is meager, hence $\bigcup_{n<\omega} A_n \in \mathcal{A}$.

Let $A \in \mathcal{A}$. Take some open U such that $U \triangle A$ is meager. We have $(X \setminus U) \triangle (X \setminus A) = U \triangle A$.

Claim 2.6.1. *If F is closed, then $F \setminus \operatorname{int}(F)$ is nwd. In particular, $F \triangle \operatorname{int}(F)$ is nwd.*

Proof of Claim 2.6.1. $F \setminus \operatorname{int}(F)$ is closed, hence $\overline{F \setminus \operatorname{int}(F)} = F \setminus \operatorname{int}(F)$. Clearly $\operatorname{int}(F \setminus \operatorname{int}(F)) = \varnothing$. □

From the claim we get that $X\backslash A =^* X\backslash U =^* \operatorname{int}(X\backslash U)$. Hence $X\backslash A \in \mathcal{A}$.

It is clear that all meager sets have the Baire property.

Let $A \in \mathcal{A}$. Then $A = (A\backslash U) \cup (A \cap U)$ for some open U such that $A\backslash U$ is meager. We have $A \cap U = U\backslash(U\backslash A)$. Thus we get that $\mathcal{A}$ is the minimal σ-algebra containing all meager and all open sets. $\square$

Theorem 2.7 (Baire Category theorem)**.** Let X be a completely metrizable space. Then every comeager set of X is dense in X.

Proof (copy from some other lecture)

Theorem and Definition 2.8. Let X be a topological space. The following are equivalent:

(i) Every nonempty open set is non-meager in X.

(ii) Every comeager set is dense.

(iii) The intersection of countable many open dense sets is dense.

In this case X is called a **Baire space**.[a]

[a] cf. Sheet 5, Exercise 1 (A.5.1)

Proof. (i) $\implies$ (ii) Consider a comeager set A. Let $U \neq \varnothing$ be any open set. Since U is non-meager, we have $A \cap U \neq \varnothing$.

(ii) $\implies$ (iii) The complement of an open dense set is nwd. Hence the intersection of countable many open dense sets is comeager.

(iii) $\implies$ (i) Let us first show that X is non-meager. Suppose that X is meager. Then $X = \bigcup_n A_n = \bigcup_n \overline{A_n}$ is the countable union of nwd sets. We have that

$$\varnothing = \bigcap_n (X\backslash\overline{A_n})$$

is dense by (iii). This proof can be adapted to other open sets X. $\square$

Notation 2.8.15. Let X, Y be topological spaces, $A \subseteq X \times Y$ and $x \in X, y \in Y$.

Let
$$A_x := \{y \in Y : (x, y) \in A\}$$
and
$$A^y := \{x \in X : (x, y) \in A\}.$$

The following similar to Fubini, but for meager sets:

Theorem 2.9 (Kuratowski-Ulam)**.** Let X, Y be second countable topological spaces. Let $A \subseteq X \times Y$ be a set with the Baire property.[a]

Then

(i) $\{x \in X : A_x$ has the BP $\}$ is comeager[b] and similarly for y.

(ii) A is meager

$$\begin{aligned} &\iff \{x \in X : A_x \text{ is meager}\} \text{ is comeager} \\ &\iff \{y \in Y : A^y \text{ is meager}\} \text{ is comeager.} \end{aligned}$$

(iii) A is comeager

$$\begin{aligned} &\iff \{x \in X : A_x \text{ is comeager}\} \text{ is comeager} \\ &\iff \{y \in Y : A^y \text{ is comeager}\} \text{ is comeager.} \end{aligned}$$

[a] It is important that A has the Baire property (cf. Sheet 5, Exercise 4 (A.5.4)).

[b] Note that not necessarily all sections have the BP. For example $\{x\} \times Y$ is meager in $X \times Y$

Proof of Theorem 2.9. (ii) and (iii) are equivalent by passing to the complement.

Claim 2.9.1. *If $F \overset{closed}{\subseteq} X \times Y$ is nwd, then*

$$\{x \in X : F_x \textit{is nwd}\}$$

is comeager.

Proof of Claim 2.9.1. Put $W = F^c$. This is open and dense in $X \times Y$. It suffices to show that $\{x \in X : W_x$ is dense$\}$ is comeager. Note that W_x is open for all x.

Fix a countable basis (V_n) of Y with V_n non-empty. We want to show that

$$\{x \in X : \forall n.\ (W_x \cap V_n) \neq \emptyset\}$$

is a comeager set. This is equivalent to

$$\{x \in X : (W_x \cap V_n) \neq \emptyset\}$$

being comeager for all n, because the intersection of countably many comeager sets is comeager.

Fix n and let $U_n := \{x \in X : (W_x \cap V_n) = \emptyset\}$. We will show that U_n is open and dense, hence it is comeager.

$U_n = \mathrm{proj}_x(W \cap (X \times V_n))$ is open since projections of open sets are open.

Let $U \subseteq X$ be nonempty and open. We need to show that $U \cap U_n \neq \emptyset$. It is

$$U \cap U_n = \mathrm{proj}_x(W \cap (U \times V_n))$$

nonempty since W is dense. □

Claim 2.9.2. *If $F \subseteq X \times Y$ is nwd, then*

$$\{x \in X : F_x \textit{is nwd}\}$$

is comeager.

Proof of Claim 2.9.2. We have that $\overline{F}$ is nwd. Hence by Claim 2.9.1 the set

$$\{x \in X : \overline{F_x} \text{ is nwd}\} \subseteq \{x \in X : F_x \text{ is nwd}\}$$

is comeager. □

Claim 2.9.3. *If $M \subseteq X \times Y$ is meager, then*

$$\{x \in X : M_x \textit{ is meager}\}$$

is comeager.

Proof of Claim 2.9.3. This follows from Claim 2.9.2: Let $M = \bigcup_{n<\omega} F_n$ where the F_n are nwd. Apply Claim 2.9.2 to each F_n. We get that M_x is comeager as a countable intersection of comeager sets. □

[Lecture 06, 2023-11-03]

(i) Let A be a set with the Baire property. Write $A = U \triangle M$ for U open and M meager. Then for all x, we have that $A_x = U_x \triangle M_x$, where U_x is open, and $\{x : M_x$ is meager$\}$ is comeager. Therefore $\{x : U_x$ open $\wedge$ M_x meager $\}$ is comeager, and for those x, A_x has the Baire property.

Claim 2.9.4. *For $P \subseteq X$, $Q \subseteq Y$ with the Baire property, let $R := P \times Q$. Then R is meager iff at least one of P or Q is meager.*

Proof of Claim 2.9.4. Suppose that R is meager. Then by Claim 2.9.3, we have that $C = \{x : R_x$ is meager $\}$ is comeager.

- If P is meager, the statement holds trivially.
- If P is not meager, then $P \cap C \neq \emptyset$. For $x \in P \cap C$ we have that R_x is meager and $R_x = Q$, hence Q is meager.

On the other hand suppose that P is meager. Then $P = \bigcup_n F_n$ for nwd sets F_n. Note that $F_n \times Y$ is nwd. So $F_n \times Q$ is also nwd. Hence $P \times Q$ is a countable union of nwd sets, so it is meager. □

(ii) " $\Longleftarrow$ " Let A be a set with the Baire property such that $\{x : A_x \text{ is meager}\}$ is comeager. Let $A = U \triangle M$ for U open and M meager. Towards a contradiction suppose that A is not meager. Then U is not meager. Since $X \times Y$ is second countable, we have that U is a countable union of open rectangles. At least one of them, say $G \times H \subseteq U$, is not meager. By Claim 2.9.4, both G and H are not meager. Since $\{x\colon A_x \text{ is meager} \wedge M_x \text{ is meager}\}$ is comeager (using Claim 2.9.3), there is $x_0 \in G$ such that A_{x_0} is meager and M_{x_0} is meager. But then H is meager as

$$H \setminus M_{x_0} \subseteq U_{x_0} \setminus M_{x_0} \subseteq U_{x_0} \triangle M_{x_0} = A_{x_0}$$

and M_{x_0} is meager $\lightning$.

" $\Longrightarrow$ " This is Claim 2.9.3.

$\square$

Remark 2.9.16. Suppose that A has the BP. Then there is an open U such that $A \triangle U =: M$ is meager. Then $A = U \triangle M$.

3 Borel sets

Definition 3.1. Let X be a topological space. Let $\mathcal{B}(X)$ denote the smallest σ-algebra, that contains all open sets. Elements of $\mathcal{B}(X)$ are called **Borel sets**.

Remark 3.1.17. Note that all Borel sets have the Baire property.

3.1 The hierarchy of Borel sets

Let ω_1 be the first uncountable ordinal. For every $d < \omega_1$, we define by transfinite recursion classes Σ^0_α and Π^0_α (or $\Sigma^0_\alpha(X)$ and $\Pi^0_\alpha(X)$ for a topological space X).

Let X be a topological space. Then define

$$\Sigma^0_1(X) := \{U \overset{\text{open}}{\subseteq} X\},$$

$$\Pi^0_\alpha(X) := \neg\Sigma^0_\alpha(X) := \{X \setminus A | A \in \Sigma^0_\alpha(X)\},$$

and for $\alpha > 1$

$$\Sigma^0_\alpha := \{\bigcup_{n<\omega} A_n : A_n \in \Pi^0_{\alpha_n}(X) \text{ for some } \alpha_n < \alpha\}.$$

Note that Π^0_1 is the set of closed sets, $\Sigma^0_2 = F_\sigma$, and $\Pi^0_2 = G_\delta$.

Furthermore define
$$\Delta^0_\alpha(X) := \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X),$$
i.e. Δ^0_1 is the set of clopen sets.

$$\begin{tikzcd}
& \underset{\text{open}}{\Sigma^0_1} \arrow[dr, "\subseteq"] & & \underset{F_\sigma}{\Sigma^0_2} \arrow[dr, "\subseteq"] & & & & \Sigma^0_\xi \arrow[dr, "\subseteq"] & & \\
\underset{\text{clopen}}{\Delta^0_1} \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \Delta^0_2 \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \Delta^0_3 & \cdots & \Delta^0_\xi \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \Delta^0_{\xi+1} & \cdots \\
& \underset{\text{closed}}{\Pi^0_1} \arrow[ur, "\subseteq"'] & & \underset{G_\delta}{\Pi^0_2} \arrow[ur, "\subseteq"'] & & & & \Pi^0_\xi \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

Proposition 3.2. Let X be a metrizable space. Then

(a) $\Sigma^0_\eta(X) \cup \Pi^0_\eta(X) \subseteq \Delta^0_\xi(X)$ for all $1 \leqslant \eta < \xi < \omega_1$.

(b) $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \Delta^0_\alpha(X)$.

Proof. (a)

Observe. *For all $1 \leqslant \alpha < \beta < \omega_1$, we have $\Pi^0_\alpha(X) \subseteq \Sigma^0_\beta(X)$ by taking "unions" of singleton sets.*

Furthermore $\Sigma^0_\alpha(X) \subseteq \Pi^0_\beta(X)$ by passing to complements.

It suffices to show $\Sigma^0_\eta(X) \subseteq \Delta^0_\xi(X)$, since $\Delta^0_\eta(X)$ is closed under complements.

Furthermore, it suffices to show $\Sigma^0_\eta(X) \subseteq \Sigma^0_\xi(X)$, by the observation (since $\Sigma^0_\eta(X) \subseteq \Pi^0_\xi(X)$ and $\Delta^0_\xi(X) = \Sigma^0_\xi(X) \cap \Pi^0_\xi(X)$).

So to prove (a) it suffices to show that for all $1 \leqslant \eta < \xi < \omega_1$, we have $\Sigma^0_\eta(X) \subseteq \Sigma^0_\xi(X)$. For $\eta = 1, \xi = 2$ this holds, since every open set is F_σ.[4]

For $\eta > 1, \xi > \eta$, we have

$$\begin{aligned}
\Sigma^0_\eta(X) &= \{\bigcup_n A_n : A_n \in \Pi^0_{\alpha_n}(X), \alpha_n < \eta\} \\
&\subseteq \{\bigcup_n B_n : B_n \in \Pi^0_{\beta_n}(X), \beta_n < \xi\} = \Sigma^0_\xi(X).
\end{aligned}$$

(b) Let $\mathcal{B}_0 := \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \Delta^0_\alpha(X)$. We need to show that $\mathcal{B}_0 = \mathcal{B}(X)$. Clearly $\mathcal{B}_0 \subseteq \mathcal{B}(X)$. It suffices to notice that $\mathcal{B}_0$ is a σ-algebra containing all open sets. Consider $\bigcup_{n<\omega} A_n$ for some $A_n \in B_0$. Then $A_n \in \Pi^0_{\alpha_n}(X)$ for some $\alpha_n < \omega_1$. Let $\alpha = \sup \alpha_n < \omega_1$. Then $\bigcup_{n<\omega} A_n \in \Sigma^0_\alpha(X)$. It is clear that $\mathcal{B}_0$ is closed under complements.

$\square$

[4]Here we use that X is metrizable!

Example 3.3. Consider the cofinite topology on ω_1. Then the non-empty open sets of this are not F_σ.

[Lecture 07, 2023-11-07]

Proposition 3.4. Let X be second countable. Then $|\mathcal{B}(X)| \leqslant \mathfrak{c}$.

Proof. We use strong induction on $\xi < \omega_1$. We have $\Sigma^0_1(X) \leqslant \mathfrak{c}$ (for every element of the basis, we can decide whether to use it in the union or not).

Suppose that $\forall \xi' < \xi.\ |\Sigma^0_{\xi'}(X)| \leqslant \mathfrak{c}$. Then $|\Pi^0_{\xi'}(X)| \leqslant \mathfrak{c}$. We have that

$$\Sigma^0_\xi(X) = \{\bigcup_n A_n : n \in \mathbb{N}, A_n \in \Pi^0_{\xi_n}(X), \xi_n < \xi\}.$$

Hence $|\Sigma^0_\xi(X)| \leqslant (\overbrace{\aleph_0}^{\{\xi' : \xi' < \xi\} \text{ is countable}} \cdot \underbrace{\mathfrak{c}}_{\text{inductive assumption}})^{\overbrace{\aleph_0}^{\text{countable unions}}}$.

We have

$$\mathcal{B}(X) = \bigcup_{\xi < \omega_1} \Sigma^0_\xi(X).$$

Hence

$$|\mathcal{B}(X)| \leqslant \omega_1 \cdot \mathfrak{c} = \mathfrak{c}.$$

□

Proposition 3.5 (Closure properties)**.** Suppose that X is metrizable. Let $1 \leqslant \xi < \omega_1$. Then

(a)
- $\Sigma^0_\xi(X)$ is closed under countable unions.
- $\Pi^0_\xi(X)$ is closed under countable intersections.
- $\Delta^0_\xi(X)$ is closed under complements.

(b)
- $\Sigma^0_\xi(X)$ is closed under *finite* intersections.
- $\Pi^0_\xi(X)$ is closed under *finite* unions.
- $\Delta^0_\xi(X)$ is closed under finite unions and finite intersections.

Proof. (a) This follows directly from the definition. Note that a countable intersection can be written as a complement of the countable union of complements:

$$\bigcap_n B_n = \left(\bigcup_n B_n^c\right)^c.$$

(b) If suffices to check this for $\Sigma^0_\xi(X)$. Let $A = \bigcup_n A_n$ for $A_n \in \Pi^0_{\xi_n}(X)$ and $B = \bigcup_m B_m$ for $B_m \in \Pi^0_{\xi'_m}(X)$. Then

$$A \cap B = \bigcup_{n,m} (A_n \cap B_m)$$

and $A_n \cap B_m \in \Pi^0_{\max(\xi_n, \xi'_m)}(X)$.

□

Example 3.6. Consider the cantor space 2^ω. We have that $\Delta^0_1(2^\omega)$ is not closed under countable unions (countable unions yield all open sets, but there are open sets that are not clopen).

3.2 Turning Borel Sets into Clopens

Theorem 3.7. [a] Let $(X, \mathcal{T})$ be a Polish space. For any Borel set $A \subseteq X$, there is a finer Polish topology,[b] such that

- A is clopen in $\mathcal{T}_A$,
- the Borel sets do not change, i.e. $\mathcal{B}(X, \mathcal{T}) = \mathcal{B}(X, \mathcal{T}_A)$.

[a] Whilst strikingly concise the verb "**to clopenize**" unfortunately seems to be non-standard vocabulary. Our tutor repeatedly advised against using it in the final exam. Contrary to popular belief the very same tutor was *not* the one first to introduce it, as it would certainly be spelled "to clopenise" if that were the case.

[b] i.e. $\mathcal{T}_A \supseteq \mathcal{T}$ and $(X, \mathcal{T}_A)$ is Polish

Corollary 3.8 (Perfect set property)**.** Let $(X, \mathcal{T})$ be Polish, and let $B \subseteq X$ be Borel and uncountable. Then there is an embedding of the cantor space 2^ω into B.

Proof. Pick $\mathcal{T}_B \supseteq \mathcal{T}$ such that $(X, \mathcal{T}_B)$ is Polish, B is clopen in $\mathcal{T}_B$ and $\mathcal{B}(X, \mathcal{T}) = \mathcal{B}(X, \mathcal{T}_B)$.

Therefore $(\mathcal{B}, \mathcal{T}_B|_B)$ is Polish. We know that there is an embedding $f\colon 2^\omega \to (B, \mathcal{T}_B|_B)$.

Consider $f\colon 2^\omega \to B \subseteq (X, \mathcal{T})$. This is still continuous as $\mathcal{T} \subseteq \mathcal{T}_B$. Since 2^ω is compact, f is an embedding. □

Proof of Theorem 3.7. We show that

$$\begin{aligned} A \coloneqq \{B \subseteq \mathcal{B}(X, \mathcal{T}) : \exists \mathcal{T}_B \supseteq \mathcal{T}. \\ &(X, \mathcal{T}_B) \text{ is Polish}, \\ &\mathcal{B}(X, \mathcal{T}) = \mathcal{B}(X, \mathcal{T}_B) \\ &B \text{ is clopen in } \mathcal{T}_B \\ \} \end{aligned}$$

is equal to the set of Borel sets. The proof rests on two lemmata:

Lemma 3.9. Let $(X, \mathcal{T})$ be a Polish space. Then for any $F \overset{\text{closed}}{\subseteq} X$ (wrt. $\mathcal{T}$) there is $\mathcal{T}_F \supseteq \mathcal{T}$ such that $\mathcal{T}_F$ is Polish, $\mathcal{B}(\mathcal{T}) = \mathcal{B}(\mathcal{T}_F)$ and F is clopen in $\mathcal{T}_F$.

Subproof. Consider $(F, \mathcal{T}|_F)$ and $(X \backslash F, \mathcal{T}|_{X \backslash F})$. Both are Polish spaces. Take the coproduct[5] $F \oplus (X \backslash F)$ of these spaces. This space is Polish, and the topology is generated by $\mathcal{T} \cup \{F\}$, hence we do not get any new Borel sets. ■

So all closed sets are in A. Furthermore A is closed under complements, since complements of clopen sets are clopen.

Lemma 3.10. Let $(X, \mathcal{T})$ be Polish. Let $\{\mathcal{T}_n\}_{n<\omega}$ be Polish topologies such that $\mathcal{T}_n \supseteq \mathcal{T}$ and $\mathcal{B}(\mathcal{T}_n) = \mathcal{B}(\mathcal{T})$. Then the topology $\mathcal{T}_\infty$ generated by $\bigcup_n \mathcal{T}_n$ is Polish and $\mathcal{B}(\mathcal{T}_\infty) = \mathcal{B}(T)$.

Proof of Lemma 3.10. We have that $\mathcal{T}_\infty$ is the smallest topology containing all $\mathcal{T}_n$. To get $\mathcal{T}_\infty$ consider

$$\mathcal{F} \coloneqq \{A_1 \cap A_2 \cap \ldots \cap A_n : A_i \in \mathcal{T}_i\}.$$

Then

$$\mathcal{T}_\infty = \{\bigcup_{i<\omega} B_i : B_i \in \mathcal{F}\}.$$

(It suffices to take countable unions, since we may assume that the $A_1, \ldots, A_n$ in the definition of $\mathcal{F}$ belong to a countable basis of the respective $\mathcal{T}_n$).

Let $Y = \prod_{n \in \mathbb{N}} (X, \mathcal{T}_n)$. Then Y is Polish. Let $\delta \colon (X, \mathcal{T}_\infty) \to Y$ defined by $\delta(x) = (x, x, x, \ldots)$.

Claim 3.10.1. *δ is a homeomorphism.*

[5] In the lecture, this was called the **topological sum**.

Subproof. Clearly δ is a bijection. We need to show that it is continuous and open.

Let $U \in \mathcal{T}_i$. Then

$$\delta^{-1}(D \cap (X \times X \times \ldots \times U \times \ldots))) = U \in \mathcal{T}_i \subseteq \mathcal{T}_\infty,$$

hence δ is continuous. Let $U \in \mathcal{T}_\infty$. Then U is the union of sets of the form

$$V = U_{n_1} \cap U_{n_2} \cap \ldots \cap U_{n_u}$$

for some $n_1 < n_2 < \ldots < n_u$ and $U_{n_i} \in \mathcal{T}_i$.

Thus is suffices to consider sets of this form. We have that

$$\delta(V) = D \cap (X \times X \times \ldots \times U_{n_1} \times \ldots \times U_{n_2} \times \ldots \times U_{n_u} \times X \times \ldots) \overset{\text{open}}{\subseteq} D.$$

■

Claim 3.10.2. $D = \{(x, x, \ldots) \in Y : x \in X\} \overset{closed}{\subseteq} Y.$

Subproof. Let $(x_n) \in Y \setminus D$. Then there are $i < j$ such that $x_i \neq x_j$. Take disjoint open $x_i \in U$, $x_j \in V$. Then

$$(x_n) \in X \times X \times \ldots \times U \times \ldots \times X \times \ldots \times V \times X \times \ldots$$

is open in $Y \setminus D$. Hence $Y \setminus D$ is open, thus D is closed. ■

It follows that D is Polish. □

We need to show that A is closed under countable unions. By Lemma 3.10 there exists a topology $\mathcal{T}_\infty$ such that $A = \bigcup_{n<\omega} A_n$ is open in $\mathcal{T}_\infty$ and $\mathcal{B}(\mathcal{T}_\infty) = \mathcal{B}(\mathcal{T})$. Applying Lemma 3.9 yields a topology $\mathcal{T}'_\infty$ such that $(X, \mathcal{T}'_\infty)$ is Polish, $\mathcal{B}(\mathcal{T}'_\infty) = \mathcal{B}(\mathcal{T})$ and A is clopen in $\mathcal{T}'_\infty$. □

[Lecture 08, 2023-11-10][6]

3.3 Parametrizations

Let Γ denote a collection of sets in some space. For us Γ will be one of $\Sigma^0_\xi(X), \Pi^0_\xi(X), \Delta^0_\xi(X), \mathcal{B}(X)$, where X is a metrizable, usually second countable space.

> **Definition 3.11.** We say that $\mathcal{U} \subseteq Y \times X$ is **Y-universal** for $\Gamma(X)$ / $\mathcal{U}$ **parametrizes** $\Gamma(X)$ iff:
>
> - $\mathcal{U} \in \Gamma(Y \times X)$,

[6] In the beginning of the lecture, we finished the proof of Lemma 3.10. This has been moved to the notes on lecture 7.

- $\{U_y : y \in Y\} = \Gamma(X)$.

Example 3.12. Let $X = \omega^\omega$, $Y = 2^\omega$ and consider $\Gamma = \Sigma^0_{\omega+5}(\omega^\omega)$. We will show that there is a 2^ω-universal set for Γ.

Theorem 3.13. Let X be a separable, metrizable space. Then for every $\xi \geqslant 1$, there is a 2^ω-universal set for $\Sigma^0_\xi(X)$ and similarly for $\Pi^0_\xi(X)$.

Proof. Note that if $\mathcal{U}$ is 2^ω universal for $\Sigma^0_\xi(X)$, then $(2^\omega \times X) \setminus \mathcal{U}$ is 2^ω-universal for $\Pi^0_\xi(X)$. Thus it suffices to consider $\Sigma^0_\xi(X)$.

First let $\xi = 1$. We construct $\mathcal{U} \overset{\text{open}}{\subseteq} 2^\omega \times X$ such that

$$\{U_y : y \in 2^\omega\} = \Sigma^0_1(X).$$

Let (V_n) be a basis of open sets of X. For all $y \in 2^\omega$ and $x \in X$ put $(y, x) \in \mathcal{U}$ iff $x \in \bigcup\{V_n : y_n = 1\}$. $\mathcal{U}$ is open. For any $V \overset{\text{open}}{\subseteq} X$, define $y \in 2^\omega$ by $y_n = 1$ iff $V_n \subseteq V$. Then $\mathcal{U}_y = V$.

Now suppose that there exists a 2^ω-universal set for $\Sigma^0_\eta(X)$ for all $\eta < \xi$. Fix $\xi_0 \leqslant \xi_1 \leqslant \ldots < \xi$ such that $\xi_n \to \xi$ if ξ is a limit, or $\xi_n = \xi'$ if $\xi = \xi' + 1$ is a successor.

Recall that $\eta_1 \leqslant \eta_2 \implies \Pi^0_{\eta_1}(X) \subseteq \Pi^0_{\eta_2}(X)$.

Note that if $A = \bigcup_n A_n$, with $A_n \in \Pi^0_{\eta_n}(X)$ for some $\eta_n < \xi$, we also have $A = \bigcup_n A'_n$ with $A'_n \in \Pi^0_{\xi_n}(X)$.

We construct a $(2^{\omega\times\omega}) \cong 2^\omega$-universal set for $\Sigma^0_\xi(X)$. For $(y_{m,n}) \in (2^{\omega\times\omega})$ and $x \in X$ we set $((y_{m,n}), x) \in \mathcal{U}$ iff $\exists n.\ ((y_{m,n})_{m<\omega}, x) \in U_{\xi_n}$, i.e. iff $\exists n.\ x \in (U_{\xi_n})_{(y_{m,n})_{m<\omega}}$.

Let $A \in \Sigma^0_\xi(X)$. Then $A = \bigcup_n B_n$ for some $B_n \in \Pi^0_{\xi_n}(X)$. Furthermore $\mathcal{U} \in \Sigma^0_\xi((2^{\omega\times\omega} \times X)$. $\square$

Remark 3.13.18. Since 2^ω embeds into any uncountable polish space Y, we can replace 2^ω by Y in the statement of the theorem.[a]

[a] By definition of the subspace topology and transfinite induction, $\Sigma^0_\xi(Y)|_{2^\omega} = \Sigma^0_\xi(2^\omega)$.

[Lecture 09, 2023-11-14]

Theorem 3.14. Let X be an uncountable Polish space. Then for all $\xi < \omega_1$, we have that $\Sigma^0_\xi(X) \neq \Pi^0_\xi(X)$.

Proof. Fix $\xi < \omega_1$. Towards a contradiction assume $\Sigma^0_\xi(X) = \Pi^0_\xi(X)$. By Theorem 3.13, there is a X-universal set $\mathcal{U}$ for $\Sigma^0_\xi(X)$.

Take $A := \{y \in X : (y, y) \notin \mathcal{U}\}$. Then $A \in \Pi^0_\xi(X)$.[7] By assumption $A \in \Sigma^0_\xi(X)$, i.e. there exists some $z \in X$ such that $A = \mathcal{U}_z$. We have

$$z \in A \iff z \in \mathcal{U}_z \iff (z, z) \in \mathcal{U}.$$

But by the definition of A, we have $z \in A \iff (z, z) \notin \mathcal{U} \lightning$. □

Definition 3.15. Let X be a Polish space. A set $A \subseteq X$ is called **analytic** iff

$$\exists Y \text{ Polish}.\ \exists B \in \mathcal{B}(Y).\ \exists \underbrace{f\colon Y \to X}_{\text{continuous}}.\ f(B) = A.$$

Trivially, every Borel set is analytic. We will see that not every analytic set is Borel.

Remark 3.15.19. In the definition we can replace the assertion that f is continuous by the weaker assertion of f being Borel.[a]

[a] use Theorem 3.7, cf. Sheet 6, Exercise 2 (A.6.2)

Theorem 3.16. Let X be Polish, $\varnothing \neq A \subseteq X$. Then the following are equivalent:

(i) A is analytic.

(ii) There exists a Polish space Y and $f\colon Y \to X$ continuous[a] such that $A = f(Y)$.

(iii) There exists $h\colon \mathcal{N} \to X$ continuous with $h(\mathcal{N}) = A$.

(iv) There is $F \overset{\text{closed}}{\subseteq} X \times \mathcal{N}$ such that $A = \operatorname{proj}_X(F)$.

(v) There is a Borel set $B \subseteq X \times Y$ for some Polish space Y, such that $A = \operatorname{proj}_X(B)$.

[a] or Borel

Proof. To show (i) $\implies$ (ii): take $B \in \mathcal{B}(Y')$ and $f\colon Y' \to X$ continuous with $f(B) = A$. Take a finer Polish topology $\mathcal{T}$ on Y' adding no Borel sets, such that B is clopen with respect to the new topology. Then let $g = f|_B$ and $Y = (B, \mathcal{T}|_B)$.

(ii) $\implies$ (iii): Any Polish space is the continuous image of $\mathcal{N}$. Let $g_1 : \mathcal{N} \to Y$ and $h := g \circ g_1$.

[7] cf. Sheet 7, Exercise 1 (A.7.1) and use that $\{(x, x) \in X^2\} \cong X$.

(iii) $\implies$ (iv): Let $h\colon \mathcal{N} \to X$ with $h(\mathcal{N}) = A$. Let $G(h) := \{(a,b) : h(a) = b\} \overset{\text{closed}}{\subseteq} \mathcal{N} \times X$ be the **graph** of h. Take $F := G(h)^{-1} := \{(c,d) | (d,c) \in G(h)\}$

Clearly (iv) $\implies$ (v).

(v) $\implies$ (i): Take $f := \mathrm{proj}_X$. □

Theorem 3.17. Let X, Y be Polish spaces. Let $f\colon X \to Y$ be **Borel** (i.e. preimages of open sets are Borel).

(a) The image of an analytic set is analytic.

(b) The preimage of an analytic set is analytic.

(c) Analytic sets are closed under countable unions and countable intersections.

Proof. (a) Let $A \subseteq X$ analytic. Then there exists Z Polish and $g\colon Z \to X$ continuous with $g(Z) = A$. We have that $f(A) = (f \circ g)(Z)$ and $f \circ g$ is Borel.

(b) Let $f\colon X \to Y$ be Borel and $B \subseteq Y$ analytic.

Take Z Polish and $B_0 \subseteq Y \times Z$ such that $\mathrm{proj}_Y(B_0) = B$. Take $f^+\colon X \times Z \to Y \times Z, f^+ = f \times \mathrm{id}$. Then

$$f^{-1}(B) = \underbrace{\mathrm{proj}_X(\overbrace{(\underbrace{f^+}_{\text{Borel}})^{-1}(\underbrace{B_0}_{\text{Borel}})}^{\text{Borel}})}_{\text{analytic}}.$$

(c) See Sheet 7, Exercise 2 (A.7.2).

□

Notation 3.17.20. Let X be Polish. Let $\Sigma_1^1(X)$ denote the set of all analytic subsets of X. $\Pi_1^1(X) := \{B \subseteq X : X \setminus B \in \Sigma_1^1(X)\}$ is the set of **coanalytic** sets.

We will see later that $\Sigma_1^1(X) \cap \Pi_1^1(X) = \mathcal{B}(X)$.

Theorem 3.18. Let X, Y be uncountable Polish spaces. There exists a Y-universal $\Sigma_1^1(X)$ set.

Proof. Take $\mathcal{U} \subseteq Y \times X \times \mathcal{N}$ which is Y-universal for $\Pi_1^0(X \times \mathcal{N})$. Let $\mathcal{V} := \mathrm{proj}_{Y \times X}(\mathcal{U})$. Then $\mathcal{V}$ is Y-universal for $\Sigma_1^1(X)$:

- $\mathcal{V} \in \Sigma_1^1(Y \times X)$ since $\mathcal{V}$ is a projection of a closed set.
- All sections of $\mathcal{V}$ are analytic. Let $A \in \Sigma_1^1(X)$. Let $C \subseteq X \times \mathcal{N}$ be closed such that $\operatorname{proj}_X(C) = A$. There is $y \in Y$ such that $\mathcal{U}_y = C$, hence $\mathcal{V}_y = A$.

□

Remark 3.18.21. In the same way that we proved $\Sigma_\xi^0(X) \neq \Pi_\xi^0(X)$ for $\xi < \omega_1$, we obtain that $\Sigma_1^1(X) \neq \Pi_1^1(X)$.

In fact if $\mathcal{U}$ is universal for $\Sigma_1^1(X)$, then $\{y : (y,y) \in \mathcal{U}\} \in \Sigma_1^1(X) \setminus \Pi_1^1(X)$. In particular, this set is not Borel.

Remark† 3.18.22. Showing that there exist sets that don't have the Baire property requires the axiom of choice. An example of such a set is constructed in Sheet 5, Exercise 4 (A.5.4).

3.4 The Lusin Separation Theorem

[Lecture 10, 2023-11-17]

Theorem 3.19 (**Lusin separation theorem**)**.** Let X be Polish and $A, B \subseteq X$ disjoint analytic. Then there is a Borel set C, such that $A \subseteq C$ and $C \cap B = \emptyset$.

Corollary 3.20. Let X be Polish. Then

$$\mathcal{B}(X) = \Delta_1^1(X),$$

where $\Delta_1^1(X) \coloneqq \Sigma_1^1(X) \cap \Pi_1^1(X)$.

Proof. Clearly $\mathcal{B}(X) \subseteq \Delta_1^1(X)$.

Let $A \in \Delta_1^1(X)$. Then $A, X \setminus A \in \Sigma_1^1(X)$. These can be separated by a Borel set C, but then $A = C$, hence $A \in \mathcal{B}(X)$. □

For the proof of the Lusin Separation Theorem (3.19), we need the following definition:

Definition 3.21. Let X be Polish, $P, Q \subseteq X$. We say that P, Q are **Borel-separable**, if there exists $R \in \mathcal{B}(X)$, such that $P \subseteq R, Q \cap R = \emptyset$.

Lemma 3.22. If $P = \bigcup_{m<\omega} P_m$, $Q = \bigcup_{n<\omega} Q_n$ are such that for any m, n the sets P_m and Q_n are Borel separable, then P and Q are Borel separable.

Proof. For all m, n pick $R_{m,n}$ Borel, such that $P_m \subseteq R_{m,n}$ and $Q_n \cap R_{m,n} = \varnothing$. Then $R = \bigcup_m \bigcap_n R_{m,n}$ has the desired property that $P \subseteq R$ and $R \cap Q = \varnothing$. $\square$

Notation 3.22.23. For $s \in \omega^{<\omega}$ be write $\mathcal{N}_s := \{x \in \mathcal{N} : x \supseteq s\}$.

Proof of Theorem 3.19. Let X be Polish, and $A, B \subseteq X$ analytic such that $A \cap B = \varnothing$ Then there are continuous surjections $f \colon \mathcal{N} \twoheadrightarrow A \subseteq X$ and $g \colon \mathcal{N} \twoheadrightarrow B \subseteq X$.

Write $A_s := f(\mathcal{N}_s)$ and $B_s := g(\mathcal{N}_s)$. Note that $A_s = \bigcup_m A_{s \frown m}$ and $B_s = \bigcup_{n<\omega} B_{s \frown n}$.

In particular $A = \bigcup_{m<\omega} A_{\underbrace{\langle m \rangle}_{\in \omega^1}}$ and $B = \bigcup_{n<\omega} B_{\langle n \rangle}$. Towards a contradiction suppose that A and B are not Borel separable. Then by Lemma 3.22, there exist m, n such that $A_{\langle m \rangle}$ and $B_{\langle n \rangle}$ can't be separated. Since $A_{\langle m \rangle} = \bigcup_i A_{\langle m,i \rangle}$ and similarly for B, there exist i, j such that $A_{\langle m,i \rangle}$ and $B_{\langle n,j \rangle}$ are not Borel separable.

Recursively, we find sequences $x, y \in \mathcal{N}$, such that $A_{x|_n}$ and $B_{y|_n}$ are not Borel separable for any $n < \omega$. So $f(x) \in A$ and $g(y) \in B$.

Recall that $A_{x|_n} = f(\mathcal{N}_{x|_n}$ and $B_{y|_n} = g(\mathcal{N}_{x|_n})$.

Since $A \cap B = \varnothing$, we get that $f(x) \neq g(y)$. Let U, V be disjoint open such that $f(x) \in U, g(y) \in V$. As f and g are continuous, $U \subseteq f(\mathcal{N}_{x|_n})$ and $V \subseteq g(\mathcal{N}_{x|_n})$ for n large enough. Then U separates $A_{x|_{n_0}}$ and V separates $B_{y|_{n_0}}$, contradicting the choice of x and y. $\square$

Theorem 3.23 (Lusin-Souslin)**.** Let X, Y be Polish and $f \colon X \to Y$ Borel. Let $A \in \mathcal{B}(X)$ such that $f|_A$ is injective. Then $f(A)$ is Borel.

Proof of Theorem 3.23. W.l.o.g. suppose that f is continuous, A is closed[8] and $X = \mathcal{N}$ by Theorem 1.24:

$$\begin{tikzcd}
\mathcal{N} & & \\
Z \arrow[u] \arrow[r, leftrightarrow, "h"] & X \arrow[r, "f"] & Y \\
h^{-1}(A) \arrow[u, hook, "\subseteq"] \arrow[r, leftrightarrow, "h|_A"'] & A \arrow[u, hook, "\subseteq"] \arrow[r, hook, "f|_A"'] & f(A) \arrow[u, hook, "\subseteq"]
\end{tikzcd}$$

[8] We might even assume that A is clopen, but we only need closed.

For $s \in \omega^{<\omega}$ write $B_s := f(\mathcal{N}_s \cap A)$.

As in the previous proof we have $B_\varnothing = f(A)$ and $B_s = \bigcup_{n<\omega} B_{s^\frown n}$ for every $s \in \omega^{<\omega}$.

Note that

- $\forall n.\ \forall s.\ B_{s^\frown n} \subseteq B_s$ and
- $\forall n \neq n'.\ \forall s.\ B_{s^\frown n} \cap B_{s^\frown n'} = \varnothing$.

The second point follows from injectivity of f and the fact that $\mathcal{N}_{s^\frown n} \cap \mathcal{N}_{s^\frown n'} = \varnothing$. In particular, the (B_s) form a Lusin scheme.

Note that $f(A) = \bigcup_{s\in\omega^k} B_s$ for every $k < \omega$, thus $f(A) = \bigcap_{k<\omega} \bigcup_{s\in\omega^k} B_s$. We want to find $B_s^* \in \mathcal{B}(X)$ for $s \in \omega^{<\omega}$, such that the B_s^* form a Lusin scheme and still

$$f(A) = \bigcap_{k<\omega} \bigcup_{s\in\omega^{<\omega}} B_s^*.$$

The existence of such B_s^* implies that $f(A)$ is Borel.

By the Corollary of the Lusin Separation Theorem (3.24), for all $k < \omega$, we can separate the collection of disjoint analytic sets $\{B_s : s \in \omega^k\}$ Borel sets, i.e. there are disjoint Borel sets $(C_s)_{s\in\omega^k}$ such that $B_s \subseteq C_s$.

Using this, we get a Lusin scheme $(B'_s)_{s\in\omega^{<\omega}}$ such that the B'_s are Borel, $B'_\varnothing = Y$ and $B_s \subseteq B'_s$: Set $B'_\varnothing = Y$ and $B'_{s^\frown n} = B'_s \cap C_{s^\frown n}$. However the B'_s might be to large.

We define another Lusin scheme $(B_s^*)_s$ as follows: Let $B_\varnothing^* := Y$, and for $s \in \omega^{<\omega}$, $n < \omega$

$$B^*_{s^\frown n} = B'_{s^\frown n} \cap \overline{B_{s^\frown n}} \cap B_s^*.$$

[Lecture 11, 2023-11-21]

Continuation of proof of Theorem 3.23. Note that $B_{(n_0,\ldots,n_k)} \subseteq B^*_{(n_0,\ldots,n_k)} \subseteq \overline{B_{n_0,\ldots,n_k}}$.

We want to show that

$$f(A) = \bigcap_{k<\omega} \bigcup_{s\in\omega^k} B_s^*.$$

Let $x \in f(A)$. Then take $a \in A$ such that $x = f(a)$. Then

$$x \in \bigcap_k \underbrace{B_{a|_k}}_{=f(A\cap N_{a|_k})} \subseteq \bigcap_k B^*_{a|_k}.$$

This gives $f(A) \subseteq \bigcap_{k<\omega} \bigcup_{s\in\omega^k} B_s^*$.

If $x \in \bigcap_{k<\omega} \bigcup_{s\in\omega^k} B_s^*$, Then there is a unique a such that $x \in \bigcap_k B^*_{a|_k}$.

Claim 3.23.1. $a \in A$.

Subproof. We have $B^*_{a|_k} \subseteq \overline{B_{a|_k}}$. So $x \in \bigcap_k \overline{B_{a|_k}}$. In particular, $B_{a|_k} \neq \varnothing$ for all k. So for all k we get that $A \cap N_{a|_k} \neq \varnothing$. But A is closed and $N_{a|_k}$ is clopen for all k. We have $\{a\} = \bigcup_k N_{a|_k}$, so $a \in A$. ■

Claim 3.23.2. $f(a) = x$.

Subproof. We have $f(a) \in \bigcap_k B_{a|_k}$. Suppose $f(a) \neq x$. Pick $U \ni f(a)$ open such that $x \notin \overline{U}$. By continuity of f, we get that $f(N_{a|_{k_0}}) \subseteq U$ for k_0 large enough. So $x \notin \overline{f(N_{a|_{k_0}})}$. In particular $x \notin \overline{f(N_{a|_{k_0}})} = \overline{B_{a|_{k_0}}} \supseteq B^*_{a|_{k_0}}$. But $x \in \bigcap_k B^*_{a|_k}$ ↯. ■

□

Corollary 3.24 (of the Lusin Separation Theorem (3.19))**.** Let X be Polish. Let $A_1, A_2, A_3, \ldots \subseteq X$ be analytic and pairwise disjoint. Then there are pairwise disjoint Borel sets $B_i \supseteq A_i$.

Proof. For all i, let B_i, C_i be disjoint Borel sets, such that $A_i \subseteq B_i$ and $\bigcup_{j \neq i} A_j \subseteq C_i$. Take $D_i := B_i \cap \bigcap_{j \neq i} C_j$. □

Theorem 3.25 (**Borel Schröder-Bernstein**)**.** Let A, B be Borel in some Polish spaces. Suppose that there are Borel embeddings $f\colon A \hookrightarrow B$ and $g\colon B \hookrightarrow A$. Then A and B are Borel isomorphic.

Proof. Cf. subsubsection A.7.4. □

Theorem 3.26 (**Isomorphism Theorem**)**.** Let X, Y be Borel in some Polish spaces. Then X is Borel isomorphic to Y iff $|X| = |Y|$.

Proof. $\Longrightarrow$ is clear. Suppose that $|X| = |Y| \leqslant \aleph_0$, then any bijection suffices, since all subsets are Borel. If $|X| = |Y| > \aleph_0$, then they must have cardinality $\mathfrak{c}$, since we can embed the Cantor space.

It suffices to show that if X is an uncountable Polish space and $\mathcal{C} = 2^\omega$ the Cantor space, then they are Borel isomorphic. There is $2^\omega \hookrightarrow X$ Borel (continuous wrt. the topology of X) On the other hand

$$X \hookrightarrow \mathcal{N} \overset{\text{continuous embedding}^9}{\hookrightarrow} \mathcal{C}$$

[9] cf. subsubsection A.2.4

For the first inclusion, recall that there is a continuous bijection $b\colon D \to X$, where $D \overset{\text{closed}}{\subseteq} \mathcal{N}$. Consider b^{-1}. Whenever $B \subseteq X$ is Borel, we have that $b^{-1}(B)$ is Borel, since b is continuous. For $A \subseteq D$ Borel be get by Lusin-Souslin (3.23), that b with respect to $b(A)$ is Borel, since $b|_A$ is injective.

Hence Schröder-Bernstein for Borel sets (3.25) can be applied. $\square$

3.5 The Projective Hierarchy

$$\begin{tikzcd}
 & \Sigma^1_1(X) \arrow[dr, hook, "\subseteq"] & & \Sigma^1_2(X) \\
\Delta^1_1(X) \arrow[ur, hook, "\subseteq"] \arrow[dr, hook, "\subseteq"'] & & \Delta^1_2(X) \arrow[ur, hook, "\subseteq"] \arrow[dr, hook, "\subseteq"] & \\
 & \Pi^1_1(X) \arrow[ur, hook, "\subseteq"'] & & \Pi^1_2(X)
\end{tikzcd}$$

Definition 3.27. Let X be a Polish space. We define

$$\begin{aligned}
\Delta^1_n(X) &:= \Sigma^1_n(X) \cap \Pi^1_n(X) \\
\Pi^1_n(X) &= \{A \subseteq X : X \setminus A \in \Sigma^1_n(X)\} \\
\Sigma^1_{n+1}(X) &= \{A \subseteq X : \exists B \in \Pi^1_n(X \times \mathcal{N}).\ A = \operatorname{proj}_X[B]\}
\end{aligned}$$

Theorem 3.28. Every analytic and every coanalytic set has the Baire property.

We will not proof this in this lecture.

3.6 Ill-Founded Trees

Recall that a **tree** on $\mathbb{N}$ is a subset of $\mathbb{N}^{<\mathbb{N}}$ closed under taking initial segments.

We now identify trees with their characteristic functions, i.e. we want to associate a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$

$$\begin{aligned}
\mathbb{1}_T\colon \omega^{<\omega} &\longrightarrow \{0,1\} \\
x &\longmapsto \begin{cases} 1 & : x \in T, \\ 0 & : x \notin T. \end{cases}
\end{aligned}$$

Note that $\mathbb{1}_T \in \{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$.

Let $\mathrm{Tr} := \{T \in 2^{\mathbb{N}^{<\mathbb{N}}} : T \text{ is a tree}\} \subseteq 2^{\mathbb{N}^{<\mathbb{N}}}$.

Observe. $\mathrm{Tr} \subseteq 2^{\mathbb{N}^{<\mathbb{N}}}$ *is closed (where we take the topology of the Cantor space).*

Indeed, for any $s \in \mathbb{N}^{<\mathbb{N}}$ we have that $\{T \in 2^{\mathbb{N}^{<\mathbb{N}}} : s \in T\}$ and $\{T \in 2^{\mathbb{N}^{<\mathbb{N}}} : s \notin T\}$ are clopen. Boolean combinations of such sets are clopen as well. In particular for s fixed, we have that

$$\{A \in 2^{\mathbb{N}^{<\mathbb{N}}} : s \in A \text{ and } s' \in A \text{ for any initial segment } s' \subseteq s\}$$

is clopen in $2^{\mathbb{N}^{<\mathbb{N}}}$.

[Lecture 12, 2023-11-24]

Definition 3.29. A tree T is **ill-founded** if it has an infinite branch, i.e. $[T] \neq \varnothing$ Otherwise it is called **well-founded**. Let

$$\mathrm{IF} := \{T \in \mathrm{Tr} : T \text{ is ill-founded}\}$$

and

$$\mathrm{WF} := \{T \in \mathrm{Tr} : T \text{ is well-founded}\}$$

Proposition 3.30. $\mathrm{IF} \in \Sigma_1^1(\mathrm{Tr})$.

Proof. We have

$$T \in \mathrm{IF} \iff \exists \beta \in \mathcal{N}.\ \forall n \in \mathbb{N}.\ T(\beta|_n) = 1.$$

Consider

$$D := \{(T, \beta) \in \mathrm{Tr} \times \mathcal{N} : \forall n.\ T(\beta|_n) = 1\}.$$

Note that this set is closed in $\mathrm{Tr} \times \mathcal{N}$, since it is a countable intersection of clopen sets. Then $\mathrm{IF} = \mathrm{proj}_{\mathrm{Tr}}(D) \in \Sigma_1^1$. ☐

Definition 3.31. An analytic set B in some Polish space Y is **complete analytic** (**Σ_1^1-complete**) iff for any analytic $A \in \Sigma_1^1(X)$ for some Polish space X, there exists a Borel function $f\colon X \to Y$ such that $x \in A \iff f(x) \in B$, i.e. $f^{-1}(B) = A$.

Similarly, a conalytic set B is called **complete coanalytic** (**Π_1^1-complete**) iff for any $A \subseteq \Pi_1^1(X)$ there exists $f\colon X \to Y$ Borel such that $f^{-1}(B) = A$.

Observe.

- *Complements of Σ_1^1-complete sets are Π_1^1-complete.*
- *Σ_1^1-complete sets are never Borel: Suppose there is a Σ_1^1-complete set $B \in \mathcal{B}(Y)$. Take $A \in \Sigma_1^1(X) \setminus \mathcal{B}(X)$*[10] *and $f\colon X \to Y$ Borel. But then we get that $f^{-1}(B)$ is Borel* ↯.

[10] e.g. Theorem 3.18

Theorem 3.32. Suppose that $A \subseteq \mathcal{N}$ is analytic. Then there is a continuous function $f \colon \mathcal{N} \to \mathrm{Tr}$ such that $x \in A \iff f(x)$ is ill-founded, i.e. $A = f^{-1}(\mathrm{IF})$.

For the proof we need some prerequisites:

Recall that for S countable, the pruned[11] trees $T \subseteq S^{<\mathbb{N}}$ on S correspond to closed subsets of $S^{\mathbb{N}}$:[12]

$$\begin{aligned} T &\longmapsto [T] \\ \{\alpha|_n : \alpha \in D, n \in \mathbb{N}\} &\longleftarrow D \end{aligned}$$

Definition 3.33. If T is a tree on $\mathbb{N} \times \mathbb{N}$ and $x \in \mathcal{N}$, then the **section at x** denoted $T(x)$, is the following tree on $\mathbb{N}$:

$$T(x) = \{s \in \mathbb{N}^{<\mathbb{N}} : (x|_{|s|}, s) \in T\}.$$

Proposition 3.34. Let $A \subseteq \mathcal{N}$. The following are equivalent:

- A is analytic.
- There is a pruned tree on $\mathbb{N} \times \mathbb{N}$ such that

$$A = \mathrm{proj}_1([T]) = \{x \in \mathcal{N} : \exists y \in \mathcal{N}.\ (x, y) \in [T]\}.$$

Proof. A is analytic iff there exists $F \overset{\text{closed}}{\subseteq} (\mathbb{N} \times \mathbb{N})^{\mathbb{N}}$ such that $A = \mathrm{proj}_1(F)$. But closed sets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ correspond to pruned trees, by the first point. $\square$

Proof of Theorem 3.32. Take a tree T on $\mathbb{N} \times \mathbb{N}$ as in Proposition 3.34, i.e. $A = \mathrm{proj}_1([T])$. Consider

$$\begin{aligned} f \colon \mathcal{N} &\longrightarrow \mathrm{Tr} \\ x &\longmapsto T(x). \end{aligned}$$

Clearly $x \in A \iff f(x) \in \mathrm{IF}$. f is continuous: Let $x|_n = y|_n$ for some $n \in \mathbb{N}$. Then for all $m \leqslant n, s, t \in \mathbb{N}^{<\mathbb{N}}$ such that $s = x|_m = y|_m$ and $|t| = |s|$, we have

- $t \in T(x) \iff (s, t) \in T$,
- $t \in T(y) \iff (s, t) \in T$.

So if $x|_n = y|_n$, then $t \in T(x) \iff t \in T(y)$ as long as $|t| \leqslant n$. $\square$

[11] no maximal elements, in particular this implies ill-founded if the tree is non empty.
[12] cf. Sheet 3, Exercise 1 (A.3.1) (c)

Corollary 3.35. IF is Σ_1^1-complete.

Proof. Let X be Polish. Suppose that $A \subseteq X$ is analytic and uncountable.

Then

$$\begin{tikzcd} X \arrow[r, "b"] & \mathcal{N} \arrow[r, "f"] & \mathrm{Tr} \\ A \arrow[u, hook] & b(A) \arrow[u, hook] \end{tikzcd}$$

where f is chosen as in Theorem 3.32.

If X is Polish and countable and $A \subseteq X$ analytic, just consider

$$\begin{aligned} g\colon X &\longrightarrow \mathrm{Tr} \\ x &\longmapsto \begin{cases} a & : x \in A, \\ b & : x \notin A, \end{cases} \end{aligned}$$

where $a \in \mathrm{IF}$ and $b \notin \mathrm{IF}$ are chosen arbitrarily. □

3.7 Linear Orders

Let us consider the space

$$\mathrm{LO} := \{x \in 2^{\mathbb{N}\times\mathbb{N}} : x \text{ is a linear order on } \mathbb{N}\},$$

where we code a linear order $(\mathbb{N}, <)$ by $x \in 2^{\mathbb{N}\times\mathbb{N}}$ with $x(m,n) = 1 \iff m \leqslant n$.

Let

$$\mathrm{WO} := \{x \in \mathrm{LO} : x \text{ is a well ordering}\}.$$

Recall that

- $(A, <)$ is a well ordering iff there are no infinite descending chains.
- Every well ordering is isomorphic to an ordinal.
- Any two well orderings are comparable, i.e. they are isomorphic, or one is isomorphic to an initial segment of the other.

  Let $(A, <_A) \prec (B, <_B)$ denote that $(A, <_A)$ is isomorphic to a proper initial segment of $(B, <_B)$.

Definition 3.36. A **rank** on some set C is a function

$$\varphi\colon C \to \mathrm{Ord}\,.$$

Example 3.37. Let $C = \mathrm{WO}$ and

$$\varphi\colon \mathrm{WO} \longrightarrow \mathrm{Ord}$$

where $\varphi((A, <_A))$ is the unique ordinal isomorphic to $(A, <_A)$.

[Lecture 13, 2023-11-08]

$\mathrm{LO} = \{x \in 2^{\mathbb{N}\times\mathbb{N}} : x \text{ is a linear order}\}$. $\mathrm{LO} \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is closed and $\mathrm{WO} = \{x \in \mathrm{LO} : x \text{ is a wellordering}\}$ is coanalytic in LO.

Another way to code linear orders:

Consider $(\mathbb{Q}, <)$, the rationals with the usual order. We can view $2^{\mathbb{Q}}$ as the space of linear orders embeddable into $\mathbb{Q}$, by associating a function $f\colon \mathbb{Q} \to \{0,1\}$ with $(f^{-1}(\{1\}), <)$.

Lemma 3.38. Any countable wellorder embeds into $(\mathbb{Q}, <)$.

Proof. [13] Cf. [Wof]. □

Definition 3.39 (**Kleene-Brouwer ordering**)**.** Let $(A, <)$ be a linear order and A countable. We define the linear order $<_{KB}$ on $A^{<\mathbb{N}}$ as follows: Let

$$s = (s_0, \ldots, s_{m-1}), t = (t_0, \ldots, t_{n-1}).$$

We set $s < t$ iff

- $(s \supsetneq t)$ or
- $s_i < t_i$ for the minimal i such that $s_i \neq t_i$.

Proposition 3.40. Suppose that $(A, <)$ is a countable well ordering. Then for a tree $T \subseteq A^{<\mathbb{N}}$ on A, Then T is well-founded iff $(T, <_{KB}|_T)$ is well ordered.

Proof. If T is ill-founded and $x \in [T]$, then for all n, we have $x|_{n+1} <_{KB} x|_n$. Thus $(T, <_{KB}|_T)$ is not well ordered.

Conversely, let $<|_{KB}$ be not a well-ordering on T. Let $s_0 >_{KB} s_1 >_{KB} s_2 >_{KB} \ldots$ be an infinite descending chain. We have that $s_0(0) \geqslant s_1(0) \geqslant s_2(0) \geqslant \ldots$ stabilizes for $n > n_0$. Let $a_0 := s_{n_0}(0)$. Now for $n \geqslant n_0$ we have that $s_n(0)$ is constant, hence for $n > n_0$ the value $s_n(1)$ must be defined. Thus there is $n_1 \geqslant n_0$ such that $s_n(1)$ is constant for all $n \geqslant n_1$. Let $a_1 := s_{n_1}(1)$ and so on. Then $(a_0, a_1, a_2, \ldots) \in [T]$. □

[13] In the lecture this was only done for countable *ordinals*.

Theorem 3.41 (Lusin-Sierpinski)**.** The set $\mathrm{LO} \setminus \mathrm{WO}$ (resp. $2^{\mathbb{Q}} \setminus \mathrm{WO}$) is Σ^1_1-complete.

Proof. We will find a continuous function $f\colon \mathrm{Tr} \to \mathrm{LO}$ such that

$$x \in \mathrm{WF} \iff f(x) \in \mathrm{WO}$$

(equivalently $x \in \mathrm{IF} \iff f(x) \in \mathrm{LO} \setminus \mathrm{WO}$). This suffices, since $\mathrm{IF} \subseteq \mathrm{Tr}$ is Σ^1_1-complete (see Corollary 3.35).

Fix a bijection $b\colon \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$.

Idea. *For* $T \in \mathrm{Tr}$ *consider* $<_{KB} |_T$.

Let $\alpha \in \mathrm{Tr}$. For $m, n \in \mathbb{N}$ define $f(\alpha)(m,n) := 1$ (i.e. $m \leqslant_{f(\alpha)} n$) iff

- $\alpha(b(m)) = \alpha(b(n)) = 1$ and $b(m) \leqslant_{KB} b(n)$ (recall that we identified Tr with a subset of $2^{\mathbb{N}^{<\mathbb{N}}}$), or
- $\alpha(b(m)) = 1$ and $\alpha(b(n)) = 0$ or
- $\alpha(b(m)) = \alpha(b(n)) = 0$ and $m \leqslant n$.

Then $\alpha \in \mathrm{WF} \iff f(\alpha) \in \mathrm{WO}$ and f is continuous. □

3.8 Π^1_1-ranks

Recall that a **rank** on a set C is a map $\varphi\colon C \to \mathrm{Ord}$.

Example 3.42.

$$\begin{aligned} \mathrm{otp}\colon \mathrm{WO} &\longrightarrow \mathrm{Ord} \\ x &\longmapsto \text{the unique } \alpha \in \mathrm{Ord} \text{ such that } x \cong \alpha. \end{aligned}$$

Definition 3.43. A **prewellordering** $\preceq$ on a set C is a binary relation that is

- reflexive,
- transitive,
- total (any two x, y are comparable),
- $\prec$ ($x \prec y \iff x \preceq y \land y \npreceq x$) is well-founded, in the sense that there are no descending infinite chains.

Remark 3.43.24.

- A prewellordering may not be a linear order since it is not necessarily antisymmetric.

- Modding out $x \sim y :\iff x \preceq y \wedge y \preceq x$ turns a prewellordering into a wellordering.

We have the following correspondence between downwards-closed ranks and prewellorderings:

$$\begin{aligned} \text{ranks} &\longrightarrow \text{prewellorderings} \\ (\varphi\colon C \to \mathrm{Ord}) &\longmapsto (x \leqslant_\varphi y :\iff \varphi(x) \leqslant \varphi(y), x, y \in C) \\ \varphi_\preceq &\longleftarrow\!\shortmid \preceq, \end{aligned}$$

where $\varphi_\preceq(x)$ is defined as

$$\begin{aligned} \varphi_\preceq(x) &:= 0 \text{ if } x \text{ is minimal}, \\ \varphi_\preceq(x) &:= \sup\{\varphi_\preceq(y) + 1 : y \prec x\}, \end{aligned}$$

i.e.

$$\varphi_\preceq(x) = \operatorname{otp}\left(\{y \in C : y \prec x\}/\sim\right).$$

Definition 3.44. Let X be Polish and $C \subseteq X$ coanalytic. Then $\varphi\colon C \to \mathrm{Ord}$ is a **Π_1^1-rank** provided that $\leqslant_\varphi^*$ and $<_\varphi^*$ are coanalytic subsets of $X \times X$, where $x \leqslant_\varphi^* y$ iff

- $y \in X \setminus C \wedge x \in C$ or
- $x, y \in C \wedge \varphi(x) \leqslant \varphi(y)$

and similarly for $<_\varphi^*$.

[Lecture 14, 2023-12-01]

Theorem 3.45 (Moschovakis)**.** If C is coanalytic, then there exists a Π_1^1-rank on C.

Proof. Pick a Π_1^1-complete set. It suffices to show that there is a rank on it. Then use the reduction to transfer it to any coanalytic set, i.e. for $x, y \in C'$ let

$$x \leqslant_{C'}^* y :\iff f(x) \leqslant_C^* f(y)$$

and similarly for $<^*$. Let $X = 2^{\mathbb{Q}} \supseteq \mathrm{WO}$. We have already shown that WO is Π_1^1-complete.

Set $\varphi(x) := \operatorname{otp}(x)$ ($\operatorname{otp}\colon \mathrm{WO} \to \mathrm{Ord}$ denotes the order type). We show that this is a Π_1^1-rank.

Define $E \subseteq \mathbb{Q}^{\mathbb{Q}} \times 2^{\mathbb{Q}} \times 2^{\mathbb{Q}}$ by

$$\begin{aligned} & (f, x, y) \in E \\ :\iff & f \text{ order embeds } (x, \leqslant_{\mathbb{Q}}) \text{ to } (y, \leqslant_{\mathbb{Q}}) \\ \iff & \forall p, q \in \mathbb{Q}.\ (p, q \in x \wedge p <_{\mathbb{Q}} q \implies f(p), f(q) \in y \wedge f(p) <_{\mathbb{Q}} f(q)) \end{aligned}$$

E is Borel as a countable intersection of clopen sets.

Define $x <_\varphi^* y$ iff

- $(x, <_\mathbb{Q})$ is well ordered and
- $(y, <_\mathbb{Q})$ does not order embed into $(x, <_\mathbb{Q})$,

where we identify $2^\mathbb{Q}$ and the powerset of $\mathbb{Q}$. This is equivalent to

- $x \in \mathrm{WO}$ and
- $\forall f \in \mathbb{Q}^\mathbb{Q}.\ (f, y, x) \notin E$,

so it is Π_1^1.[14]

Furthermore $x \leqslant_\varphi^* y \iff$ either $x <_\varphi^* y$ or $(x, <_\mathbb{Q})$ and $(y, <_\mathbb{Q})$ are well ordered with the same order type, i.e. either $x <_\varphi^* y$ or $x, y \in \mathrm{WO}$ and any order embedding of $(x, <_\mathbb{Q})$ to $(y, <_\mathbb{Q})$ is cofinal[15] in $(y, <_\mathbb{Q})$ and vice versa. Equivalently, either $(x <_\varphi^* y)$ or

$$\begin{aligned} & x, y \in \mathrm{WO} \\ \wedge\ & \forall f \in \mathbb{Q}^\mathbb{Q}.\ (E(f, x, y) \implies \forall p \in y.\ \exists q \in x.\ p \leqslant f(q)) \\ \wedge\ & \forall f \in \mathbb{Q}^\mathbb{Q}.\ (E(f, y, x) \implies \forall p \in x.\ \exists q \in y.\ p \leqslant f(q)) \end{aligned}$$

□

Theorem 3.46. Let X be Polish and $R \subseteq X \times \mathbb{N}$ by Π_1^1 (we only need that $\mathbb{N}$ is countable). Then there is $R^* \subseteq R$ coanalytic such that

$$\forall x \in X.\ (\exists n.\ (x, n) \in R \iff \exists! n.\ (x, n) \in R^*).$$

We say that R^* **uniformizes** R.[a]

[a] Wikimedia has a nice picture.

Proof. Let $\varphi\colon R \to \mathrm{Ord}$ be a Π_1^1-rank. Set

$$\begin{aligned} (x, n) \in R^* :\iff\ & (x, n) \in R \\ & \wedge \forall m.\ (x, n) \leqslant_\varphi^* (x, m) \\ & \wedge \forall m.\ \big((x, n) <_\varphi^* (x, m) \vee n \leqslant m\big), \end{aligned}$$

i.e. take the element with minimal rank that has the minimal second coordinate among those elements. □

[14] (very informal) Note that Σ_1^1-sets work well with comprehensions using "∃": Writing $A \in \Sigma_1^1(X)$ as $A = \mathrm{proj}_X(B)$ for some Borel set $B \subseteq X \times Y$, the second coordinate can be thought of as being a witness for a statement. Likewise, being complements of Σ_1^1-sets, Π_1^1-sets can capture that a witness does not exist, i.e. they interact nicely with "∀".

[15] Recall that $A \subseteq (x, <_\mathbb{Q})$ is **cofinal** if $\forall t \in x.\ \exists a \in A.\ t \leqslant_\mathbb{Q} a$.

Remark 3.46.25. Uniformization also works for $R \in \Pi^1_1(X \times Y)$ for arbitrary Polish spaces X, Y, cf. [Kec12, (36.12)].

Corollary 3.47 (Countable Reduction for Π^1_1 Sets)**.** Let X be a Polish space and $(C_n)_n$ a sequence of coanalytic subsets of X.

Then there exists a sequence (C_n^*) of pairwise disjoint Π^1_1-sets with $C_n^* \subseteq C_n$ and
$$\bigcup_{n \in \mathbb{N}} C_n^* = \bigcup_{n \in \mathbb{N}} C_n.$$

Proof. Define $R \subseteq X \times \mathbb{N}$ by setting $(x, n) \in R :\iff x \in C_n$ and apply Theorem 3.46. □

Let X be a Polish space. If $(X, \prec)$ is well-founded (i.e. there are no infinite descending chains) then we define a rank $\rho_\prec \colon X \to \mathrm{Ord}$ as follows: For minimal elements the rank is 0. Otherwise set $\rho_\prec(x) := \sup\{\rho_\prec(y) + 1 : y \prec x\}$. Let $\rho(\prec) := \sup\{\rho_\prec(x) + 1 : x \in X\}$.

Fact 3.47.26 ([Kec12, Appendix B])**.** Since $\rho_\prec \colon X \to \rho(\prec)$ is surjective, we have that $\rho(\prec) < |X|^+$.[a]

[a] Here, $|X|^+$ denotes the successor cardinal.

Theorem 3.48 (Kunen-Martin, [Kec12, (31.1)])**.** If $(X, \prec)$ is well-founded and $\prec \subseteq X^2$ is Σ^1_1 then $\rho(\prec) < \omega_1$.

Proof. Wlog. $X = \mathcal{N}$. There is a tree S on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ (i.e. $S \subseteq (\mathbb{N} \times \mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$) such that
$$\forall x, y \in \mathcal{N}.\ (x \succ y \iff \exists \alpha \in \mathcal{N}.\ (x, \alpha, y) \in [S]).$$ [16]

Let
$$W := \{w = (s_0, u_1, s_1, \ldots, u_n, s_n) : s_i, u_i \in \mathbb{N}^n \wedge (s_{i-1}, u_i, s_i) \in S\}.$$

Clearly $|W| \leqslant \aleph_0$. Define $\prec^*$ on W by setting
$$(s_0, u_1, s_1, \ldots, u_n, s_n) \succ^* (s'_0, u'_1, s'_1, \ldots, u'_m, s'_m)$$

iff

- $n < m$ and

[16] Here we use that $\prec$ is analytic, i.e. $\prec$ can be written as the projection of a closed subset of $(\mathcal{N} \times \mathcal{N}) \times \mathcal{N}$ and closed subsets correspond to pruned trees.

- $\forall i \leqslant n.\ s_i \subsetneq s'_i \wedge u_i \subsetneq u'_i$.

Claim 1. $\prec^*$ *is well-founded.*

Subproof. If $w_n = (s_0^n, u_1^n, \ldots, u_n^n, s_n^n)$ was descending, then let

$$x_i := \bigcup s_i^n \in \mathcal{N}$$

and

$$\alpha_i := \bigcup_n u_i^n \in \mathcal{N}.$$

We get $(x_{i-1}, \alpha_i, x_i) \in [S]$, hence $x_{i-1} \succ x_i$ for all i, but this is an infinite descending chain in the original relation $\lightning$ ■

Hence $\rho(\prec^*) < |W|^+ \leqslant \omega_1$. We can turn $(X, \prec)$ into a tree $(T_\prec, \subsetneq)$ with

$$\rho(\prec) = \rho(T_\prec)$$

by setting $\varnothing \in T_\prec$ and $(x_0, \ldots, x_n) \in T_\prec$,$x_i \in X = \mathcal{N}$, iff $x_0 \succ x_1 \succ x_2 \succ \ldots \succ x_n$.

For all $x \succ y$ pick $\alpha_{x,y} \in \mathcal{N}$ such that $(x, \alpha_{x,y}, y) \in [S]$. Define

$$\begin{aligned} \varphi \colon T_\prec \backslash \{\varnothing\} &\longrightarrow W \\ (x_0, x_1, \ldots, x_n) &\longmapsto (x_0|_n, \alpha_{x_0,x_1}|_n, x_1|_n, \ldots, \alpha_{x_{n-1},x_n}|_n, x_n|_n). \end{aligned}$$

Then φ is a homomorphism of $\supsetneq$ to $\prec^*$ so

$$\rho(\prec) = \rho(T_\prec \backslash \{\varnothing\}, \supsetneq) \leqslant \rho(\prec^*) < \omega_1.$$

□

[Lecture 15, 2023-12-05]

Theorem 3.49 (Boundedness Theorem)**.** Let X be Polish, $C \subseteq X$ coanalytic, $\varphi \colon C \to \omega_1$ a coanalytic rank on C, $A \subseteq C$ analytic, i.e. $A \in \Sigma_1^1(X)$. Then $\sup\{\varphi(x) : x \in A\} < \omega_1$.

Moreover for all $\xi < \omega_1$,

$$D_\xi := \{x \in C : \varphi(x) < \xi\}$$

and

$$E_\xi := \{x \in C : \varphi(x) \leqslant \xi\}$$

are Borel subsets of X.

Proof. Let

$$\begin{aligned} x \prec y :&\iff x, y \in A \wedge \varphi(x) < \varphi(y) \\ &\iff x, y \in A \wedge y \nleqslant_{\varphi}^{*} x. \end{aligned}$$

Since A is analytic, this relation is analytic and wellfounded on X. By Kunen-Martin (3.48) we get $\rho(\prec) < \omega_1$. Thus $\sup\{\varphi(x) : x \in A\} < \omega_1$.

Since $D_\xi = \bigcup_{\eta < \xi} E_\xi$, it suffices to check $E_\xi \in \Sigma_1^1(X)$. Let $\alpha := \sup\{\varphi(x) : x \in C\}$. Then $E_\xi = E_\alpha$ for all $\alpha \leqslant \xi < \omega_1$.

Consider $\xi \leqslant \alpha$.

- If there exists $x_0 \in C$ with $\varphi(x_0) \geqslant \xi$, pick such x_0 of minimal rank. Then for all $y \in X$ we have

$$\begin{aligned} y \in E_\xi &\iff y \in C \wedge \varphi(y) \leqslant \xi \\ &\iff y \leqslant_{\varphi}^{*} x_0 && \text{coanalytic} \\ &\iff x_0 \nless_{\varphi}^{*} y && \text{analytic} \end{aligned}$$

  So E_ξ is Borel.

- If there exists no such x_0 then $\xi = \alpha$ and

$$E_\xi = E_\alpha = \bigcup_{\eta < \alpha} E_\eta$$

  is a countable union of Borel sets by the previous case.

$\square$

4 Abstract Topological Dynamics

Recall:

Definition† 4.0.27. Let X be a set. A **group action** of a group G on X is a function $\alpha\colon G \times X \to X$ such that

- $\forall x \in X.\ \alpha(1_G, x) = x$,
- $\forall g, h \in G, x \in X.\ \alpha(gh, x) = \alpha(g, \alpha(h, x))$.

Often we will abbreviate $\alpha(g, x)$ as $g \cdot x$.

For $x \in X$, the **orbit** of x is defined as

$$G \cdot x := \{g \cdot x : g \in G\}.$$

A group action is called **transitive** iff $g \mapsto g \cdot x$ is surjective for all $x \in X$, i.e. iff the action has exactly one orbit.

For $x \in X$, the **stabilizer subgroup** of G with respect to x is

$$G_x := \{g \in G : g \cdot x = x\}.$$

Remark† 4.0.28. Group actions of a group G on a set X correspond to group homomorphisms $G \to \mathrm{Sym}(X)$. Indeed for a group action $\alpha\colon G \times X \to X$ consider

$$\begin{aligned} G &\longrightarrow \mathrm{Sym}(X) \\ g &\longmapsto (x \mapsto g \cdot x). \end{aligned}$$

Definition† 4.0.29. A group G with a topology is a **topological group** iff

$$\begin{aligned} G \times G &\longrightarrow G \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned}$$

are continuous.

Definition 4.1. Let T be a topological group[a] and let X be a compact metrizable space.

A **flow** (X, T), sometimes denoted $T \curvearrowright X$ is a continuous action

$$\begin{aligned} T \times X &\longrightarrow X \\ (t, x) &\longmapsto tx. \end{aligned}$$

A flow is **minimal** iff every orbit is dense.

(Y, T) is a **subflow** of (X, T) if $Y \subseteq X$ and Y is invariant under T, i.e. $\forall t \in T, y \in Y.\ ty \in Y$.

A flow (X, T) is **isometric** iff there is a metric d on X such that for all $t \in T$ the map

$$\begin{aligned} a_t \colon X &\longrightarrow X \\ x &\longmapsto tx \end{aligned}$$

is an **isometry**, i.e. $\forall t \in T.\ \forall x, y \in X.\ d(a_t(x), a_t(y)) = d(x, y)$.

If (X, T) is a flow, then a pair (x, y), $x \neq y$ is **proximal** iff

$$\exists z \in X.\ \exists (t_n)_{n<} \in T^\omega.\ t_n x \xrightarrow{n \to \infty} z \wedge t_n y \xrightarrow{n \to \infty} z.$$

A flow is **distal** iff it has no proximal pair.

[a] usually $T = \mathbb{Z}$ with the discrete topology

Remark 4.1.30. Note that a flow is minimal iff it has no proper subflows.

Definition† 4.1.31. Let (T, X) and (T, Y) be flows. A **factor map** $\pi\colon (T, X) \to (T, Y)$ is a continuous surjection $X \twoheadrightarrow Y$ that is T-equivariant, i.e. $\forall t \in T, x \in X.\ \pi(t \cdot x) = t \cdot \pi(x)$. If such a factor map exists, we also say that (T, Y) is a **factor** of (T, X).

An **isomorphism** from (T, X) to (T, Y) is a homeomorphism $X \leftrightarrow Y$ commuting with the group action.

Warning† 4.1.32. What is called "factor" here is called "subflow" by Furstenberg.

Example 4.2. Recall that $S_1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $X = S_1$, $T = S_1$ $(\alpha, \beta) \mapsto \alpha + \beta$ is isometric.[a]

[a] Note that here we consider the abelian group structure of S^1 and $\alpha + \beta$ denotes the addition of *angles*, i.e. $\alpha \cdot \beta$ in complex numbers.

Definition 4.3. Let X, Y be compact metric spaces and $\pi\colon (X,T) \to (Y,T)$ a factor map. Then (X,T) is an **isometric extension** of (Y,T) if there is $\rho\colon X \times_Y X \to \mathbb{R}$[a] such that

(a) ρ is continuous.

(b) For each $y \in Y$, ρ is a metric on the fiber $X_y := \{x \in X : \pi(x) = y\}$.

(c) $\forall t \in T.\ \rho(tx_1, tx_2) = \rho(x_1, x_2)$.

(d) $\forall y, y' \in Y.$ the metric spaces (X_y, ρ) and $(X_{y'}, \rho)$ are isometric.

[a] Recall that in the category of topological spaces the **fiber product** of $A \xrightarrow{f} C$, $B \xrightarrow{g} C$ is $A \times_C B = \{(a,b) \in A \times B : f(a) = g(b)\}$, i.e. $X \times_Y X = \{(x_1, x_2) \in X^2 : \pi(x_1) = \pi(x_2)\}$.

Remark 4.3.33. A flow is isometric iff it is an isometric extension of the trivial flow, i.e. the flow acting on a singleton. Indeed maps $\rho\colon X \times_\star X = X^2 \to \mathbb{R}$ as in Definition 4.3 correspond to metrics witnessing that the flow is isometric.

Proposition 4.4. An isometric extension of a distal flow is distal.

Proof. Let $\pi\colon X \to Y$ be an isometric extension. Towards a contradiction, suppose that $x_1, x_2 \in X$ are proximal. Take $z \in X$ and a sequence $(g_n)_{n<\omega}$ in T such that $g_n x_1 \to z$ and $g_n x_2 \to z$.

Then $g_n\pi(x_1) \to \pi(z)$ and $g_n\pi(x_2) \to \pi(z)$, so by distality of Y we have $\pi(x_1) = \pi(x_2)$. Then $\rho(g_n x_1, g_n x_2)$ is defined and equal to $\rho(x_1, x_2)$. By the continuity of ρ, we get $\rho(g_n x_1, g_n x_2) \to \rho(z,z) = 0$. Therefore $\rho(x_1, x_2) = 0$. Hence $x_1 = x_2$ $\lightning$. $\square$

Definition 4.5. Let $\Sigma = \{(X_i, T) : i \in I\}$ be a collection of factors of (X,T). Let $\pi_i\colon (X,T) \to (X_i, T)$ denote the factor map. Then (X,T) is a **limit**[a] of Σ iff

$$\forall x_1 \neq x_2 \in X.\ \exists i \in I.\ \pi_i(x_1) \neq \pi_i(x_2).$$

[a] This is not a limit in the category theory sense and not uniquely determined.

Proposition 4.6. A limit of distal flows is distal.

Proof. Let (X,T) be a limit of $\Sigma = \{(X_i, T) : i \in I\}$. Suppose that each (X_i, T) is distal. If (X,T) was not distal, then there were $x_1, x_2, z \in X$ and a sequence (g_n) in T with $g_n x_1 \to z$ and $g_n x_2 \to z$. Take $i \in I$ such that $\pi_i(x_1) \neq \pi_i(x_2)$.

But then $g_n \pi_i(x_1) \to \pi_i(z)$ and $g_n \pi_i(x_2) \to \pi_i(z)$, which is a contradiction since (X_i, T) is distal. □

[Lecture 16, 2023-12-08]

X is always compact metrizable.

Theorem 4.7. Every minimal isometric flow $(X, \mathbb{Z})$ for X a compact metrizable space[a] is isomorphic to an abelian group rotation $(K, \mathbb{Z})$, with K an abelian compact group and some fixed $\alpha \in K$ such that $h(x) = x + \alpha$ for all $x \in K$

[a]Such a flow is uniquely determined by $h \colon X \to X, x \mapsto 1 \cdot x$.

Proof. The action of 1 determines h. Consider

$$\{h^n : n \in \mathbb{Z}\} \subseteq \mathcal{C}(X, X) = \{f \colon X \to X : f \text{ continuous}\},$$

where the topology is the uniform convergence topology. Let $G = \overline{\{h^n : n \in \mathbb{Z}\}} \subseteq \mathcal{C}(X, X)$. Since the family $\{h^n : n \in \mathbb{Z}\}$ is uniformly equicontinuous, i.e.

$$\forall \varepsilon > 0.\ \exists \delta > 0.\ d(x, y) < \delta \implies \forall n.\ d(h^n(x), h^n(y)) < \varepsilon,$$

we have by the Arzelà-Ascoli-Theorem that G is compact.

G is a closure of a topological group, hence it is a topological group, by Fact B.0.86. Since h^n and h^m commute for all $n, m \in \mathbb{Z}$, we obtain that G is abelian.

Take any $x \in X$ and consider the orbit $G \cdot x$. Since $\mathbb{Z} \curvearrowright X$ is minimal, i.e. every orbit is dense, we have that $G \cdot x$ is dense in X.

Claim 1. *$G \cdot x$ is compact.*

Subproof. Since $\mathbb{Z} \curvearrowright X$ is continuous, $g \mapsto gx$ is continuous:

Let g_n be a sequence in G such that $g_n \to g$. Then $g_n x \to gx$, since the topology on $\mathcal{C}(X, X)$ is the uniform convergence topology.

Therefore the compactness of G implies that the orbit Gx is compact. ■

Since $G \cdot x$ is compact and dense, we get $G \cdot x = X$ (compact subsets of Hausdorff spaces are closed).

Let $G_x = \{f \in G : f(x) = x\} < G$ be the stabilizer subgroup. Note that $G_x \subseteq G$ is closed. Take $K := {}^{G}\!/\!_{G_x}$ with the quotient topology.

There is a continuous bijection

$$\begin{aligned} K &\longrightarrow X \\ fG_x &\longmapsto f(x). \end{aligned}$$

By compactness this is a homeomorphism, so this is an isomorphism between flows.

For $\alpha = h$ we get that a flow $\mathbb{Z} \curvearrowright X$ corresponds to $\mathbb{Z} \curvearrowright K$ with $(1, x) \mapsto x + \alpha$. $\square$

Definition 4.8. Let (X, T) be a flow and (Y, T) a factor of (X, T). Suppose there is $\eta \in \mathrm{Ord}$ such that for any $\xi < \eta$ there is a factor (X_ξ, T) of (X, T) with factor map $\pi_\xi \colon X \to X_\xi$ such that

(a) $(X_0, T) = (Y, T)$ and $(X_\eta, T) = (X, T)$.

(b) If $\xi < \xi'$, then (X_ξ, T) is a factor of $(X_{\xi'}, T)$ "inside (X, T)", i.e. $\pi_\xi = \pi_{\xi,\xi'} \circ \pi_{\xi'}$, where $\pi_{\xi,\xi'} \colon X_{\xi'} \to X_\xi$ is the factor map.

(c) $\forall \xi < \eta$. $(X_{\xi+1}, T)$ is an isometric extension of (X_ξ, T).

(d) $\xi \leqslant \eta$ is a limit, then (X_ξ, T) is a limit of $\{(X_\alpha, T), \alpha < \xi\}$.

$$\begin{tikzcd}
X \arrow[dr, "\pi_{\xi'}"] \arrow[ddr, "\pi_\xi"'] \arrow[dddd, bend right, "\pi"'] & \\
& X_{\xi'} \arrow[d, "\pi_{\xi,\xi'}"] \\
& X_\xi \\
& \\
Y &
\end{tikzcd}$$

Then we say that (X, T) is a **quasi-isometric extension** of (Y, T).

Definition 4.9. If (Y, T) is trivial, i.e. $|Y| = 1$, then a quasi-isometric extension (X, T) of (Y, T) is called a **quasi-isometric flow**.

Corollary 4.10. Every quasi-isometric flow is distal.

Proof. The trivial flow is distal. Apply Proposition 4.4 and Proposition 4.6. $\square$

Theorem 4.11 (Furstenberg)**.** Every minimal distal flow is quasi-isometric.

By Zorn's lemma, this will follow from

Theorem 4.12 (Furstenberg)**.** Let (X, T) be a minimal distal flow and let (Y, T) be a proper factor.[a] Then there is another factor (Z, T) of (X, T) which is a proper isometric extension of Y.

$$\begin{tikzcd}
(Y,T) & & (X,T) \arrow[ll] \arrow[dl] \\
& (Z,T) \arrow[ul, "\text{isometric extension}"] &
\end{tikzcd}$$

[a] i.e. (X, T) and (Y, T) are not isomorphic

Theorem 4.11 allows us to talk about ranks of distal minimal flows:

Definition 4.13 ([Fur63, 13.1])**.** Let (X, T) be a quasi-isometric flow, and let η be the smallest ordinal such that there exists a quasi-isometric system $\{(X_\xi, T), \xi \leqslant \eta\}$ with $(X, T) = (X_\xi, T)$. Then η is called the **rank** or **order** of the flow and is denoted by $\operatorname{rank}((X, T))$.

Definition† **4.13.34.** Let X be a topological space. Let $K(X)$ denote the set of all compact subspaces of X and $K(X)^* := K(X) \backslash \{\varnothing\}$. If $d \leqslant 1$ is a metric on X, we can equip $K(X)$ with a metric d_H given by

$$\begin{aligned}
d_H(\varnothing, \varnothing) &:= 0, \\
d_H(K, \varnothing) &:= 1 & K \neq \varnothing, \\
d_H(K_0, K_1) &:= \max\{\max_{x \in K_0} d(x, K_1), \max_{x \in K_1} d(x, K_0)\} K_0, K_1 \neq \varnothing.
\end{aligned}$$

The topology induced by the metric is given by basic open subsets[a] of the form $[U_0; U_1, \ldots, U_n]$, for $U_0, \ldots, U_n \overset{\text{open}}{\subseteq} X$, where

$$[U_0; U_1, \ldots, U_n] := \{K \in K(X) | K \subseteq U_0 \wedge \forall 1 \leqslant i \leqslant n.\ K \cap U_i \neq \varnothing\}.$$

[a] cf. Sheet 9, Exercise 2 (A.9.2)

We want to view flows as a metric space. For a fixed compact metric space X, we can view the flows $(X, \mathbb{Z})$ as a subset of $\mathcal{C}(X, X)$. Note that $\mathcal{C}(X, X)$ is Polish.[17] Then the minimal flows on X are a Borel subset of $\mathcal{C}(X, X)$.[18]

However we do not want to consider only flows on a fixed space X, but we want to look all flows at the same time. The Hilbert cube $\mathbb{H} = [0, 1]^{\mathbb{N}}$ embeds all compact metric spaces. Thus we can consider $K(\mathbb{H})$, the space of compact subsets of $\mathbb{H}$. $K(\mathbb{H})$ is a Polish space.[19] Consider $K(\mathbb{H}^2)$. A flow $\mathbb{Z} \curvearrowright X$

[17] cf. Sheet 1, Exercise 4 (A.1.4)
[18] Exercise
[19] cf. Sheet 9, Exercise 2 (A.9.2), Sheet 12, Exercise 4 (A.12.4)

corresponds to the graph of

$$\begin{aligned} X &\longrightarrow X \\ x &\longmapsto 1 \cdot x \end{aligned}$$

and this graph is an element of $K(\mathbb{H}^2)$.

Theorem 4.14 (Beleznay-Foreman)**.** Consider $\mathbb{Z}$-flows.

- For any $\alpha < \omega_1$, there is a distal minimal flow of rank α.
- Distal flows form a Π^1_1-complete set, where flows are identified with their graphs as elements of $K(\mathbb{H}^2)$ as above.
- Moreover, this rank is a Π^1_1-rank.

4.1 The Ellis semigroup

[Lecture 17, 2023-12-12]

Let (X, d) be a compact metric space and (X, T) a flow.

Let $X^X := \{f : X \to X\}$ be the set of all functions.[20] We equip this with the product topology, i.e. a subbasis is given by sets

$$U_\varepsilon(x, y) := \{f \in X^X : d(x, f(y)) < \varepsilon\}.$$

for all $x, y \in X$, $\varepsilon > 0$.

X^X is a compact Hausdorff space.[21]

Remark 4.14.35. [a] Let $f_0 \in X^X$ be fixed.

- $X^X \ni f \mapsto f \circ f_0$ is continuous:

  Consider $\{f : f \circ f_0 \in U_\varepsilon(x, y)\}$. We have $f \circ f_0 \in U_\varepsilon(x, y)$ iff $f \in U_\varepsilon(x, f_0(y))$.
- Fix $x_0 \in X$. Then $f \mapsto f(x_0)$ is continuous.
- In general $f \mapsto f_0 \circ f$ is not continuous, but if f_0 is continuous, then the map is continuous.

[a] cf. Sheet 11, Exercise 1 (A.11.1)

Definition 4.15. Let (X, T) be a flow. Then the **Ellis semigroup** is defined by $E(X, T) := \overline{T} \subseteq X^X$, i.e. identify $t \in T$ with $x \mapsto tx$ and take the closure in X^X.

[20] We take all the functions, they need not be continuous.
[21] cf. Tychonoff's theorem

$E(X, T)$ is compact and Hausdorff, since X^X has these properties.

Properties of (X, T) translate to properties of $E(X, T)$:

Goal. *We want to show that if (X, T) is distal, then $E(X, T)$ is a group.*

Proposition 4.16. $E(X, T)$ is a semigroup, i.e. closed under composition.

Proof. Let $G := E(X, T)$. Take $t \in T$. We want to show that $tG \subseteq G$, i.e. for all $h \in G$ we have $th \in G$.

We have that $t^{-1}G$ is compact, since t^{-1} is continuous and G is compact.

It is $T \subseteq t^{-1}G$ since $T \ni s = t^{-1} \underbrace{(ts)}_{\in G}$.

So $G = \overline{T} \subseteq t^{-1}G$. Hence $tG \subseteq G$.

Claim 1. *If $g \in G$, then*

$$\overline{T}g = \overline{Tg}.$$

Subproof. Cf. Sheet 11, Exercise 1 (A.11.1) ■

Let $g \in G$. We need to show that $Gg \subseteq G$.

It is

$$Gg = \overline{T}g = \overline{Tg}.$$

Since G compact, and $Tg \subseteq G$, we have $\overline{Tg} \subseteq G$. □

Definition 4.17. A **compact semigroup** S is a nonempty semigroup[a] with a compact Hausdorff topology, such that $S \ni x \mapsto xs$ is continuous for all s.

[a]may not contain inverses or the identity

Example 4.18. The Ellis semigroup is a compact semigroup.

Lemma 4.19 (Ellis–Numakura)**.** Every non-empty compact semigroup contains an **idempotent** element, i.e. f such that $f^2 = f$.

Proof. Using Zorn's lemma, take a $\subseteq$-minimal compact subsemigroup R of S and let $s \in R$.

Then Rs is also a compact subsemigroup and $Rs \subseteq R$. By minimality of R, $R = Rs$. Let $P := \{x \in R : xs = s\}$. Then $P \neq \emptyset$, since $s \in Rs$ and P is a compact semigroup, since $x \overset{\alpha}{\mapsto} xs$ is continuous and $P = \alpha^{-1}(s) \cap R$. Thus $P = R$ by minimality, so $s \in P$, i.e. $s^2 = s$. □

The Ellis-Numakura Lemma (4.19) is not very interesting for $E(X,T)$, since we already know that it has an identity. But it is interesting for other semigroups.

Theorem 4.20 (Ellis)**.** (X,T) is distal iff $E(X,T)$ is a group.

Proof. Let $G := E(X,T)$ and let d be a metric on X. For all $g \in G$ we need to show that $x \mapsto gx$ is injective. If we had $gx = gy$, then $d(gx, gy) = 0$. Then $\inf_{t \in T} d(tx, ty) = 0$, but the flow is distal, hence $x = y$.

Let $g \in G$. Consider the compact semigroup $\Gamma := Gg$. By the Ellis-Numakura Lemma (4.19), there is $f \in \Gamma$ such that $f^2 = f$, i.e. for all $x \in X$ we have $f^2(x) = f(x)$. Since f is injective, we get that $x = f(x)$, i.e. $f = \mathrm{id}$.

Since $f \in Gg$, there exists $g' \in G$ such that $f = g' \circ g$.

It is $g' = g'gg'$, so $\forall x.\ g'(x) = g'(gg'(x))$. Hence g' is injective and $x = gg'(x)$, i.e. $gg' = \mathrm{id}$.

On the other hand if (x_0, x_1) is proximal, then there exists $g \in G$ such that $gx_0 = gx_1$.[22] It follows that an inverse to g can not exist. □

Theorem 4.21. If (X,T) is distal, then X is the disjoint union of minimal subflows. In fact those disjoint sets will be orbits of $E(X,T)$.

Proof. Let $G = E(X,T)$. Note that for all $x \in X$, we have that $Gx \subseteq X$ is compact and invariant under the action of G.

Since G is a group, the orbits partition X.[23]

We need to show that (Gx, T) is minimal. Suppose that $y \in Gx$, i.e. $Gx = Gy$. Since $g \mapsto gy$ is continuous, we have $Gx = Gy = \overline{T}y = \overline{Ty}$, so Ty is dense in Gx. □

Corollary 4.22. If (X,T) is distal and minimal, then $E(X,T) \curvearrowright X$ is transitive.

4.2 Sketch of proof of Theorem 4.12

[Lecture 18, 2023-12-15]

The goal for this lecture is to give a very rough sketch of Theorem 4.12 in the case of $|Z| = 1$.

Let (X,T) be a distal flow. Then $G := E(X,T)$ is a group.

[22] cf. Sheet 11, Exercise 1 (A.11.1) (e)

[23] Note that in general this does not hold for semigroups.

Definition 4.23. For $x, x' \in X$ define

$$F(x,x') := \inf\{d(gx,gx') : g \in G\}.$$

Fact 4.23.36. (a) $F(x,x') = F(x',x)$,

(b) $F(x,x') \geqslant 0$ and $F(x,x') = 0$ iff $x = x'$.

(c) $F(gx,gx') = F(x,x')$ since G is a group.

(d) F is an **upper semi-continuous**[a] function on X^2, i.e. $\forall a \in R$. $\{(x,x') \in X^2 : F(x,x') < a\} \overset{\text{open}}{\subseteq} X^2$.

This holds because F is the infimum of continuous functions

$$\begin{aligned} f_g\colon X^2 &\longrightarrow \mathbb{R} \\ (x,x') &\longmapsto d(gx,gx') \end{aligned}$$

for $g \in G$.

[a] Wikimedia has a nice picture.

Theorem and Definition 4.24. The sets

$$U_a(x) := \{x' : F(x,x') < a\}$$

form the basis of a topology in X. This topology is called the **F-topology** on X. In this setting, the original topology is also called the **E-topology**.

This will follow from the following lemma:

Lemma 4.25. Let $F(x,x') < a$. Then there exists $\varepsilon > 0$ such that whenever $F(x',x'') < \varepsilon$, then $F(x,x'') < a$.

Proof of Theorem and Definition 4.24. We have to show that if $U_a(x_1) \cap U_b(x_2) \neq \varnothing$, then this intersection is the union of sets of this kind. Let $x' \in U_a(x_1) \cap U_b(x_2)$. Then by Lemma 4.25, there exists $\varepsilon_1 > 0$ with $U_{\varepsilon_1}(x') \subseteq U_a(x_1)$. Similarly there exists $\varepsilon_2 > 0$ such that $U_{\varepsilon_2}(x') \subseteq U_b(x_2)$. So for $\varepsilon \leqslant \varepsilon_1, \varepsilon_2$, we get $U_\varepsilon(x') \subseteq U_a(x_1) \cap U_b(x_2)$. ☐

Proof of Lemma 4.25. [24][25]

Let $T = \bigcup_n T_n$,T_n compact, wlog. $T_n \subseteq T_{n+1}$, and let $G(x,x') := \{(gx,gx') : g \in G\} \subseteq X \times X$. Take b such that $F(x,x') < b < a$. Then $U = \{(u,u') \in G(x,x') : d(u,u') < b\}$ is open in $G(x,x')$ and since $F(x,x') < b$ we have $U \neq \varnothing$.

[24] Not relevant for the exam.

[25] This was not covered in class.

Claim 4.25.1. *There exists n such that*

$$\forall (u,u') \in G(x,x').\ T_n(u,u') \cap U \neq \varnothing.$$

Subproof. Suppose not. Then for all n, there is $(u_n, u_n') \in G(x,x')$ with

$$T_n(u_n, u_n') \subseteq G(x,x') \setminus U.$$

Note that the RHS is closed. For $m > n$ we have $T_n(u_m, u_m') \subseteq G(x,x') \setminus U$ since $T_n \subseteq T_m$. By compactness of X, there exists v, v' and some subsequence such that $(u_{n_k}, u_{n_k}') \to (v, v')$.

So for all n we have $T_n(v,v') \subseteq G(x,x') \setminus U$, hence $T(v,v') \cap U = \varnothing$, so $G(v,v') \cap U = \varnothing$. But this is a contradiction as $\varnothing \neq U \subseteq G(v,v')$. ■

The map

$$\begin{aligned} T \times X &\longrightarrow X \\ (t,x) &\longmapsto tx \end{aligned}$$

is continuous. Since T_n is compact, we have that $\{(x,t) \mapsto tx : t \in T_n\}$ is equicontinuous. So there is $\varepsilon > 0$ such that $d(x_1, x_2) < \varepsilon \implies d(tx_1, tx_2) < a - b$ for all $t \in T_n$. Sheet 11

Suppose now that $F(x', x'') < \varepsilon$. Then there is $t_0 \in T$ such that $d(t_0 x', t_0 x'') < \varepsilon$, hence $d(tt_0 x', tt_0 x'') < a - b$ for all $t \in T_n$. Since $(t_0 x, t_0 x') \in G(x,x')$, there is $t_1 \in T_n$ with $(t_1 t_0 x, t_1 t_0 x') \in U$, i.e. $d(t_1 t_0 x, t_1 t_0 x') < b$ and therefore $F(x, x'') = d(t_1 t_0 x, t_1 t_0 x'') < a$. □

Now assume $Z = \{\star\}$. We want to sketch a proof of Theorem 4.12 in this case, i.e. show that if (Z,T) is a proper factor of a minimal distal flow (X,T) then there is another factor (Y,T) of (X,T) which is a proper isometric extension of Z.

Proof (sketch).

1. For $x \in X$ define

$$\begin{aligned} F_x \colon X &\longrightarrow \mathbb{R} \\ x' &\longmapsto F(x,x'). \end{aligned}$$

2. Define an equivalence relation on X, by $x_1 \sim x_2 :\iff \{x \in X : F_{x_1}(x) = F_{x_2}(x)\}$ is comeager in X[26]. Then for all $g \in G$ we have $x_1 \sim x_2 \implies gx_1 \sim gx_2$.

   Let $M := \{[x]_\sim : x \in X\} = X/\sim$ bet the quotient space. It is compact, second countable and Hausdorff. Let $\pi \colon X \to M$ denote the quotient map.

[26] with respect to the E-topology

3. $(Y,T) =: (M,T)$ is an isometric flow:

   (a) For $a > 0$, $x, x' \in X$ let

   $$W(x,x') := \{g \in G : F(x,gx') < a\}.$$

   This turns out to be a subbasis of a topology which is coarser than the original topology on G. The new topology makes G compact.

   (b) Let $\theta(g)$ be the transformation of M defined by $\theta(g)\pi(x) = \pi(gx)$. This is well defined. Let $H = \theta(G)$. This is just a quotient of G, $g \mapsto \theta(g)$ may not be injective.

   (c) One can show that H is a topological group and (M,H) is a flow.[27]

   (d) Since H is compact, (M,H) is equicontinuous, i.e. it is isometric. In particular, (M,T) is isometric.

4. $M \neq \{\star\}$, i.e. (M,T) is non-trivial:

   Suppose towards a contradiction that $M = \{\star\}$, i.e. $x_1 \sim x_2$ for all $x_1, x_2 \in X$. Fix x_2. For every $x_1 \in X$ we have that

   $$\{x : F(x_1,x) = F(x_2,x)\}$$

   is comeager. Let x_1 be a point of continuity of F_{x_2}. Let $\langle a_n : n < \omega\rangle$ be a sequence of elements that set, i.e. $F(x_1,a_n) = F(x_2,a_n)$, such that $a_n \to x_1$. So by the continuity of F_{x_2} at x_1

   $$\lim_{n\to\infty} F(x_2,a_n) = F(x_2,x_1)$$

   and by the definition of F

   $$\lim_{n\to\infty} F(x_1,a_n) = F(x_1,x_1) = 0.$$

   So

   $$F(x_2,x_1) = \lim_{n\to\infty} F(x_2,a_n) = \lim_{n\to\infty} F(x_1,a_n) = 0$$

   and by distality we get $x_1 = x_2$. Since almost all points of X are points of continuity of F_{x_2} (Theorem 4.26) this implies that $X\backslash\{x_2\}$ is meager. But then $X = \{\star\}$ ↯.

□

Theorem 4.26. [a] Let X be a metric space and $\Gamma\colon X \to \mathbb{R}$ be upper semicontinuous. Then the set of continuity points of Γ is comeager.

[a] Not covered in class

[27] This is non-trivial.

Proof. [28] Take x such that Γ is not continuous at x. Then there is an $\varepsilon > 0$ and $x_n \to x$ such that $\Gamma(x_n) + \varepsilon \leqslant \Gamma(x)$. Take $q \in \mathbb{Q}$ such that $\Gamma(x) - \varepsilon < q < \Gamma(x)$. Then let

$$B_q := \{a \in X : \Gamma(a) \geqslant q\}.$$

$X \backslash B_q = \{a \in X : \Gamma(a) < q\}$ is open, i.e. B_q is closed. Note that $x \in F_q := B_q \backslash \operatorname{int}(B_q)$ and $B_q \backslash \operatorname{int}(B_q)$ is nwd as it is closed and has empty interior, so $\bigcup_{q \in \mathbb{Q}} F_q$ is meager. $\square$

4.3 The Order of a Flow

[Lecture 19, 2023-12-19]

See also [Tao08, Lecture 6].

Definition† 4.26.37. Let X, Y be metric spaces. A family F of functions $X \to Y$ is called **equicontinuous** at $x_0 \in X$ iff

$$\forall \varepsilon > 0.\ \exists \delta > 0.\ \forall f \in F.\ d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \varepsilon.$$

It is called equicontinuous iff it is equicontinuous at every point. It is called **uniformly equicontinuous** iff

$$\forall \varepsilon > 0.\ \exists \delta > 0.\ \forall x_0 \in X.\ \forall f \in F.\ d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \varepsilon.$$

A flow (X, T) is called equicontinuous iff T is equicontinuous.

Note that since X compact the notions of equicontinuity and uniform equicontinuity coincide.

Fact† 4.26.38 ([Tao08, Lecture 6, Exercise 1])**.** A flow (X, T) is isometric iff it is equicontinuous.

Proof. Clearly an isometric flow is equicontinuous. On the other hand suppose that T is uniformly equicontinuous. Define a metric $\tilde{d}$ on X by setting $\tilde{d}(x, y) := \sup_{t \in T} d(tx, ty) \leqslant 1$ (wlog. $d \leqslant 1$). By equicontinuity of T we get that $\tilde{d}$ and d induce the same topology on X. $\square$

Recall that we defined the order of a quasi-isometric flow to be the minimal number of steps required when building the tower to reach the flow with a quasi-isometric system (cf. Theorem 4.12, Definition 4.13).

Theorem 4.27 (Maximal isometric factor)**.** For every flow (X, T) there is a maximal factor (Y, T), $\pi\colon X \to Y$, i.e. if $(Y', T), \pi'\colon X \to Y'$ is any isometric factor of (X, T), then (Y', T) is a factor of (Y, T).

[28] Not relevant for the exam.

$$
\begin{tikzcd}
(X,T) \arrow[d] \arrow[r] & \begin{array}{c}(Y,T)\\ \text{maximal isometric}\end{array} \arrow[dl, dashed, "\exists"] \\
\begin{array}{c}(Y',T)\\ \text{isometric}\end{array} &
\end{tikzcd}
$$

Proof. We want to apply Zorn's lemma. If suffices to show that isometric flows are closed under inverse limits,[29] i.e. if $(Y_\alpha, f_{\alpha,\beta})$, $\beta < \alpha \leqslant \Theta$ are isometric, then the inverse limit Y is isometric.

$$
\begin{tikzcd}
 & X \arrow[d, dashed, "\pi_\alpha"] \arrow[dd, bend left, "\pi_\beta"] \\
Y \arrow[r, "f_\alpha"] \arrow[dr, "f_\beta"'] & Y_\alpha \arrow[d, dashed, "f_{\alpha,\beta}"] \\
 & Y_\beta
\end{tikzcd}
$$

Consider

$$\Delta_\alpha := \{(y, y') \in Y^2 : f_\alpha(y) = f_\alpha(y')\}.$$

Let d be a metric on Y and d_α a metric on Y_α, wlog. $d, d_\alpha \leqslant 1$. Note that $\beta < \alpha \implies \Delta_\beta \supseteq \Delta_\alpha$ and

$$\bigcap_{\alpha \leqslant \theta} \Delta_\alpha = \{(y, y) : y \in Y\}.$$

Fix $\varepsilon > 0$ and consider

$$\{(y, y') \in \Delta_\alpha : d(y, y') \geqslant \varepsilon\}.$$

By the finite intersection property we get

$$\exists \alpha.\ f_\alpha(y) = f_\alpha(y') \implies d(y, y') < \varepsilon,$$

i.e. $\forall z \in Y_\alpha.\ \operatorname{diam}(f_\alpha^{-1}(z)) \leqslant \varepsilon$.

Towards a contradiction assume that Y is not isometric, i.e. not equicontinuous. Then there are $(y_j), (y_j') \in Y$ such that $d(y_j, y_j') \to 0$ and $\varepsilon > 0, t_j \in T$ such that $d(t_j y_j, t_j y_j') > \varepsilon$.

By compactness wlog. (y_j) and (y_j') converge (to the same point). Find α such that $f_\alpha(y) = f_\alpha(y') \implies d(y, y') < \frac{\varepsilon}{4}$. Let $z_j := f_\alpha(y_j)$ and $z_j' := f_\alpha(y_j')$. Then (z_j) and (z_j') converge to the same point $z \in Y_\alpha$. By equicontinuity of (Y_α, T),

[29] This seems to be an inverse limit in the category theory sense.

$d_{Y_\alpha}(t_jz_j, t_jz'_j) \to 0$. Wlog. (t_jz_j) and $(t_jz'_j)$ converge. Let z^* be their limit. On the one hand, by the triangle inequality we get

$$d(f_\alpha^{-1}(t_jz_j), f_\alpha^{-1}(t_jz'_j)) > \underbrace{\varepsilon}_{<d(t_jy_j,t_jy'_j)} - \overbrace{\frac{\varepsilon}{4}}^{\text{Diameter of fiber}} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

On the other hand, from

$$\begin{aligned} d(f_\alpha^{-1}(t_jz_j), f_\alpha^{-1}(z^*)) &\to 0, \\ d(f_\alpha^{-1}(t_jz'_j), f_\alpha^{-1}(z^*)) &\to 0, \\ \operatorname{diam} f_\alpha^{-1}(\{z^*\}) &< \frac{\varepsilon}{4} \end{aligned}$$

we obtain

$$d(f_\alpha^{-1}(t_jz_j), f_\alpha^{-1}(t_jz'_j)) < \frac{\varepsilon}{2} \lightning.$$

□

More generally we can show:

Theorem 4.28 ([Fur63, Prop. 13.1])**.** Let (X,T) be a distal flow and $(Z,T) = \pi(X,T)$ a factor. Then there exists an isometric extension (Y,T) of (Z,T) which is a factor of (X,T), such that (Y,T) is maximal among such extensions, i.e. if (Y',T) is any flow with these two properties, then (Y',T) is a factor of (Y,T).

$$\begin{tikzcd}
\substack{(X,T)\\ \text{distal}} \arrow[dr] \arrow[ddrr, bend left] \arrow[ddd, "\pi"] & & \\
& (Y,T) \arrow[dr, dashed] \arrow[ddl, "\text{max. iso.}"] & \\
& & (Y',T) \arrow[dll, "\text{iso.}"] \\
(Z,T) & &
\end{tikzcd}$$

Such a factor (Y,T) is called a **maximal isometric extension** of (Z,T).

Lemma 4.29. Let four flows be given as in

$$\begin{tikzcd}
 & (Y,T) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
(Z_1,T) \arrow[dr, "w_1"'] & & (Z_2,T) \arrow[dl, "w_2"] \\
 & (W,T) &
\end{tikzcd}$$

Suppose that whenever $y \neq y' \in Y$, then $\pi_1(y) \neq \pi(y')$ or $\pi_2(y) \neq \pi_2(y')$.

If (Z_1, T) is an isometric extension of (W, T), then (Y, T) is an isometric extension of (Z_2, T).

Proof. For $z_1, z_1' \in Z_1$ with $w_1(z_1) = w_1(z_1')$ let $\rho(z_1, z_1')$ be the metric on the fiber of Z_1 over W. Set $\sigma(y, y') := \rho(\pi_1(y), \pi_1(y'))$ whenever $\pi_2(y) = \pi_2(y')$. In this case $w_2 \circ \pi_2(y) = w_2 \circ \pi_2(y')$ and $w_1 \circ \pi_1(y) = w_1 \circ \pi_1(y')$, so σ is well defined. σ is a semi-metric[30] on the fibers of Y over Z_2 and invariant under T.

σ is a metric on fibers, since if $\pi_2(y) = \pi_2(y')$ and $\sigma(y, y') = 0$, then $\pi_1(y) = \pi_1(y')$ or $y = y'$. $\square$

Definition 4.30. A quasi-isometric system $\{(X_\xi, T) : \xi \leqslant \eta\}$ is called **normal** if $(X_{\xi+1}, T)$ is the maximal isometric extension of (X_ξ, T) in (X_η, T) for all $\xi < \eta$.

Theorem 4.31 ([Fur63, 13.2])**.** If $\{(X_\xi, T), \xi \leqslant \eta\}$ is a normal quasi-isometric system, then (X_η, T) has order η.

Proof. We only sketch the proof here. Details can be found in [Fur63], section 13. Let $\{(X'_\xi, T), \xi \leqslant \eta'\}$ be another quasi-isometric system terminating with $(X_\eta, T) = (X'_{\eta'}, T)$. We want to show that $\eta' \geqslant \eta$. For this, we show that for all $\xi < \eta$, (X'_ξ, T) is a factor of (X_ξ, T) using transfinite induction.

[30] Like a metric, but the distinct points can have distance 0.

$$
\begin{tikzcd}
X'_{\eta'} \arrow[r, phantom, "\cdots"] \arrow[rr, bend left, "\pi'_3"] \arrow[rrr, bend left, "\pi'_2"] \arrow[rrrr, bend left, "\pi'_1"] \arrow[d, equal] & {} & X'_3 \arrow[r] & X'_2 \arrow[r] & X'_1 \\
X \arrow[d, equal] & & & & \\
X_\eta \arrow[r, phantom, "\cdots"] \arrow[rr, bend right, "\pi_3"'] \arrow[rrr, bend right, "\pi_2"'] \arrow[rrrr, bend right, "\pi_1"'] & {} & X_3 \arrow[r] \arrow[uu, dotted] & X_2 \arrow[r] \arrow[uu, dotted] & X_1 \arrow[uu, dotted]
\end{tikzcd}
$$

We'll only show the successor step:

Suppose we have $(X'_\xi, T) = \theta((X_\xi, T)$. Let π_ξ and π'_ξ denote the maps from X to X_ξ resp. X'_ξ. Set

$$
Y := \{(\pi_\xi(x), \pi'_{\xi+1}(x)) \in X_\xi \times X'_{\xi+1} : x \in X\}
$$

Then

$$
\begin{tikzcd}
(X_{\xi+1}, T) \arrow[rr, dashed] \arrow[dr, "\text{max. iso}"'] & & (Y, T) \arrow[dl, "\pi"', "\text{iso}"] \arrow[dr, "\pi'"] & \\
& (X_\xi, T) \arrow[dr, "\theta"'] & & (X'_{\xi+1}, T) \arrow[dl, "\text{iso}"', "\theta'"] \\
& & (X'_\xi, T) &
\end{tikzcd}
$$

The diagram commutes, since all maps are the induced maps. By definition of Y is clear that π and π' separate points in Y. Thus Lemma 4.29 can be applied. Since θ' is an isometric extension, so is π. Then (Y, T) is a factor of $(X_{\xi+1}, T)$ by the maximality of the isometric extension $(X_{\xi+1}, T) \to (X_\xi, T)$.

In particular, $(X'_{\xi+1}, T)$ is a factor of $(X_{\xi+1}, T)$. $\square$

Example 4.32 ([Fur63, p. 513])**.** Let X be the infinite torus

$$
X := \{(\xi_1, \xi_2, \ldots) : \xi_i \in \mathbb{C}, |\xi_i| = 1\}.
$$

Let π_n be the projection to the first n coordinates and $X_n := \pi_n(X)$.

Let $\tau_1(\xi_1, \xi_2, \ldots, \xi_n, \ldots) = (e^{\mathbf{i}\alpha}\xi_1, \xi_1\xi_2, \ldots, \xi_{n-1}\xi_n, \ldots)$ where $\frac{\alpha}{\pi}$ is irrational. Let $T = \langle \tau_1 \rangle \cong \mathbb{Z}$.

We will show that (X_n, T) is minimal for all n, and so (X, T) is minimal. Furthermore (X_{n+1}, T) is the maximal isometric extension of (X_n, T) so

(X, T) has order ω.

[Lecture 20, 2024-01-09]

Example 4.33. [a] Let $X = (S^1)^{\mathbb{N}}$[b] and consider $(X, \mathbb{Z})$ where the action is generated by

$$\tau\colon (x_1, x_2, x_3, \ldots) \mapsto (x_1 + \alpha, x_1 + x_2, x_2 + x_3, \ldots)$$

for some irrational α.

[a] This is the same as Example 4.32, but with new notation.
[b] We identify S^1 and $\mathbb{R}/\mathbb{Z}$.

Remark† 4.33.39. Note that we can identify S^1 with a subset of $\mathbb{C}$ (and use multiplication) or with $\mathbb{R}/\mathbb{Z}$ (and use addition). In the lecture both notations were used. Here I'll try to only use multiplicative notation.

We will be studying projections to the first d coordinates, i.e.

$$\tau_d\colon (x_1, \ldots, x_d) \mapsto (e^{\mathbf{i}\alpha} x_1, x_1 x_2, \ldots, x_{d-1} x_d).$$

τ_d is called the **d-skew shift**. For $d = 1$ we get the circle rotation $x \mapsto e^{\mathbf{i}\alpha} x$.

Fact 4.33.40. The circle rotation $x \mapsto e^{\mathbf{i}\alpha} x$ is minimal. In fact, every subgroup of S^1 is either dense in S^1 or it is of the form

$$H_m := \{x \in S^1 : x^m = 0\}$$

for some $m \in \mathbb{Z}$.[a]

[a] cf. Sheet 12, Exercise 2 (A.12.2)

We will show that τ_d is minimal for all d, i.e. every orbit is dense. From this it will follow that τ is minimal.

Let $\pi_n\colon X \to (S^1)^n$ be the projection to the first n coordinates.

Lemma 4.34. Let $x, x' \in X$ with $\pi_n(x) = \pi_n(x')$ for some n. Then there is a sequence of points x_k with

$$\pi_{n-1}(x_k) = \pi_{n-1}(x) = \pi_{n-1}(x')$$

for all k and

$$F(x_k, x) \xrightarrow{k\to\infty} 0, F(x_k, x') \xrightarrow{k\to\infty} 0,$$

where F is as in Definition 4.23, i.e. $F(a,b) = \inf_{n\in\mathbb{Z}} d(\tau^n a, \tau^n b)$, where d is the metric on X, $d((x_i),(y_i)) = \max_n \frac{1}{2^n}|x_n - y_n|$.

Proof of Lemma 4.34. Let

$$\begin{aligned} x &= (\alpha_1^0, \alpha_2^0, \ldots, \alpha_{n-1}^0, \alpha_n, \alpha_{n+1}, \alpha_{n+2}, \ldots) \\ x' &= (\alpha_1^0, \alpha_2^0, \ldots, \alpha_{n-1}^0, \alpha_n, \alpha'_{n+1}, \alpha'_{n+2}, \ldots). \end{aligned}$$

We will choose x_k of the form

$$(\alpha_1^0, \alpha_2^0, \ldots, \alpha_{n-1}^0, \alpha_n e^{\mathbf{i}\beta_k}, \alpha_{n+1}, \alpha_{n+2}, \ldots),$$

where β_k is such that $\frac{\beta_k}{\pi}$ is irrational and $|\beta_k| < 2^{-k}$. Fix a sequence'(b)). of such β_k. Then

$$d(x_k, x) = 2^{-n}|e^{\mathbf{i}\beta_k} - 1| < 2^{-n-k} \xrightarrow{k\to\infty} 0.$$

In particular $F(x_k, x) \to 0$.

We want to show that $F(x_k, x') < 2^{-n-k}$. For $u, u' \in X$, $u = (\xi_n)_{n\in\mathbb{N}}$, $u' = (\xi'_n)_{n\in\mathbb{N}}$, let $\frac{u}{u'} = (\frac{\xi_n}{\xi'_n})_{n\in\mathbb{N}}$ (X is a group). We are interested in $F(x_k, x') = \inf_m d(\tau^m x_k, \tau^m x')$, but it is easier to consider the distance between their quotient and 1. Consider

$$w_k := \frac{x_k}{x'} = (\underbrace{1, \ldots, 1}_{n-1}, e^{\mathbf{i}\beta_k}, \overbrace{\frac{\alpha_{n+1}}{\alpha'_{n+1}}, \frac{\alpha_{n+2}}{\alpha'_{n+2}}, \ldots}^{\text{not interesting}}).$$

Claim 4.34.1. *It is*

$$F(x_k, x') = \inf_m d(\sigma^m(w_k), 1),$$

where $\sigma(\xi_1, \xi_2, \ldots) = (\xi_1, \xi_1\xi_2, \xi_2\xi_3, \ldots)$.

Subproof. We have

$$\begin{aligned} F(u, u') &= \inf_m d(\tau^m u, \tau^m u') \\ &= \inf_m d(\frac{\tau^m u}{\tau^m u'}, 1) \\ &= \inf_m d(\sigma^m\left(\frac{u}{u'}\right), 1). \end{aligned}$$

■

Fix k. Let $w^* = (1, \ldots, 1, e^{\mathbf{i}\beta_k}, 1, \ldots)$. By minimality of (X, T) for any $\varepsilon > 0$, there exists $m \in \mathbb{Z}$ such that $d(\sigma^m w_k, w^*) < \varepsilon$. Then

$$\begin{aligned}\inf_m d(\sigma^m w_k, 1) &\leqslant \inf_m d(\sigma^m w_k, w^*) + d(w^*, 1) \\ &\leqslant 2^{-n}|e^{\mathbf{i}\beta_k} - 1| \\ &< 2^{-n-k}.\end{aligned}$$

□

Definition 4.35. For every continuous $f\colon S^1 \to S^1$, the **winding number** $[f] \in \mathbb{Z}$ is the unique integer such that f is homotopic[a] to the map $x \mapsto x^n$.

[a] $f\colon Y \to Z$ and $g\colon Y \to Z$ are homotopic iff there is $H\colon Y \times [0,1] \to \mathbb{Z}$ continuous such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$.

Remark 4.35.41. Note that for

$$\begin{aligned}\sigma\colon (S^1)^d &\longrightarrow S^1 \\ (x_1, \ldots, x_d) &\longmapsto x_d\end{aligned}$$

we have that $T = \tau_{d+1}$, where

$$\begin{aligned}T\colon (S^1)^d \times S^1 &\longrightarrow (S^1)^d \times S^1 \\ (y, x_{d+1}) &\longmapsto (\tau_d(y), \sigma(y)x_{d+1}).\end{aligned}$$

Theorem 4.36. For every d if τ_d[a] is minimal, then τ_{d+1} is minimal.

[a] more formally $((S^1)^d, \langle\tau_d\rangle)$

Corollary 4.37. τ_d is minimal for all d.

Proof. τ_1 is minimal (Fact 4.33.40). Apply Theorem 4.36. □

Corollary 4.38. Since all the τ_d are minimal, τ is minimal.

Proof. We need to show that every orbit is dense. This follows from the definition of the product topology, since for a basic open set $U = U_1 \times \ldots \times U_d \times (S^1)^\infty$ it suffices to analyze the first d coordinates. □

Proof of Theorem 4.36. Let $S := \tau_d$, $T := \tau_{d+1}$ and $Y := (S^1)^d$. Consider

$$\begin{aligned}\gamma\colon S^1 &\longrightarrow Y \\ x &\longmapsto (x, x, \ldots, x).\end{aligned}$$

Note that

(a) γ and $S \circ \gamma$ are homotopic via

$$\begin{aligned} H\colon S^1 \times [0,1] &\longrightarrow (S^1)^d \\ (x,t) &\longmapsto (xe^{\mathbf{i}t\alpha}, x^{t+1}, x^{t+1}, x^{t+1}, \ldots, x^{t+1}) \end{aligned}$$

(b) For all $m \in \mathbb{Z}\backslash\{0\}$, we have $[x \mapsto (\sigma(\gamma(x)))^m] = m \neq 0$, since $\sigma(\gamma(x)) = \sigma((x,\ldots,x)) = x$.

[Lecture 21, 2024-01-12]

Continuation of proof of Theorem 4.36. Suppose towards a contradiction that $Y \times S^1$ contains a proper minimal subflow Z. Consider the projection $\pi\colon Y \times S^1 \to Y$. By minimality of Y, we have $\pi(Z) = Y$. Note that for every $\theta \in S^1$, $\theta \cdot Z$ is minimal, so either $\theta \cdot Z = Z$ or $(\theta \cdot Z) \cap Z = \varnothing$.[31]

Let $H = \{\theta \in S^1 : \theta \cdot Z = Z\}$. H is a closed subgroup of S^1. Therefore either $H = S^1$ (but in that case $Z = Y \times S^1$, so this cannot be the case), or there exists $m \in \mathbb{Z}$ such that $H = \{\xi \in S^1 : \xi^m = 1\}$ by Fact 4.33.40.

Note that if $(y, \beta) \in Z$ then for $t \in S^1$, we have

$$(y, \beta \cdot t) \in Z \iff t^m = 1.$$

Therefore for every $y \in Y$, there are exactly m many $\xi \in S^1$ such that $(y,\xi) \in Z$.

Specifically for all y there exists $\beta^{(y)} \in S^1$ such that $(y,\xi) \in Z$ iff

$$\xi \in \{\beta^{(y)} \cdot t_1, \beta^{(y)} \cdot t_2, \ldots, \beta^{(y)} \cdot t_m\},$$

where the $t_i \in S^1$ are such that $t_i^m = 1$ for all i and $i \neq j \implies t_i \neq t_j$, i.e. the t_i are the m^{th} roots of unity.

Consider $f\colon (y,\xi) \mapsto (y, \xi^m)$. Since $(\beta^{(y)} \cdot t_i)^m = (\beta^{(y)})^m$ we get a continuous function $\varphi\colon Y \to S^1$ such that

$$Z = \{(y,\xi) \in Y \times S^1 : \xi^m = \varphi(y)\},$$

namely

$$\begin{aligned} \varphi\colon Y &\longrightarrow S^1 \\ y &\longmapsto (\beta^{(y)})^m \end{aligned}$$

Z is isomorphic to m copies of the graph of that function, hence the graph is closed, so the function is continuous.

Note that $f(Z)$ is homeomorphic to Y (for every $y \in Y$, $\varphi(y)$ is the unique element such that $(y, \varphi(y)) \in f(Z)$).

[31] actually $(1,\ldots,1,\theta) \cdot Z$, we identify S^1 and $\{0\}^d \times S^1 \subseteq Y \times S^1$.

Claim 4.36.1. $\varphi(S(y)) = \varphi(y) \cdot (\sigma(y))^m$.

Subproof. We have $T(y, \xi) = (S(y), \sigma(y) \cdot \xi)$ (cf. Remark 4.35.41). Z is invariant under T. So for $(y, \xi) \in Z$ we get $T(y, \xi) = (S(y), \sigma(y) \cdot \xi) \in Z$. Thus

$$\begin{aligned}
\varphi(S(y)) &= (\sigma(y) \cdot \xi)^m \\
&= (\sigma(y))^m \cdot \xi^m \\
&= (\sigma(y))^m \cdot \varphi(y).
\end{aligned}$$

■

Applying γ we obtain

$$[\varphi \circ S \circ \gamma] = [\varphi \circ \gamma] + [x \mapsto (\sigma(\gamma(x))^n].$$

$S \circ \gamma$ is homotopic to γ, so $[\varphi \circ S \circ \gamma] = [\varphi \circ \gamma]$. Thus $[x \mapsto (\sigma(\gamma(x))^n] = 0$, but that is a contradiction to (b) $\lightning$ □

Let $X_n := (S^1)^n$ and $X := (S^1)^{\mathbb{N}}$.

Theorem 4.39. (X_n, τ_n) is the maximal isometric extension of (X_{n-1}, τ_{n-1}) in (X, τ).

Corollary 4.40. The order of (X, τ) is ω.

[Lecture 22, 2024-01-16]

Proof of Theorem 4.39. We have the following situation:

$$\begin{tikzcd}
X \arrow[dr, "\pi_n"] \arrow[drr, bend left, "\pi'"] \arrow[ddr, bend right, "\pi_{n-1}"'] & & \\
& X_n \arrow[d, "\text{isometric}"] & Y \arrow[l, dashed, "h"'] \arrow[dl, bend left, "\overline{g}\text{, max. isom.}"] \\
& X_{n-1} &
\end{tikzcd}$$

We want to show that this tower is normal, i.e. the isometric extensions are maximal isometric extension. Let Y be a maximal isometric extension of X_{n-1} in X and let $\overline{g} = \pi^n_{n-1} \circ h$. We need to show that h is an isomorphism. Towards a contradiction assume that h is not an isomorphism. Then there are $x, x' \in X$ with $\pi'(x) \neq \pi'(x')$ but $\pi_n(x) = \pi_n(x') = t \in X_n$. Then $h^{-1}(t) \ni \pi'(x), \pi'(x')$.

By a Lemma 4.34 there is a sequence (x_k) in X with $\pi_{n-1}(x_k) = \pi_{n-1}(x) = \pi_{n-1}(x')$ for all k, such that $F(x_k, x) \to 0$ and $F(x_k, x') \to 0$.

Let ρ be a metric witnessing that $\overline{g}$ is an isometric extension, i.e. ρ is defined on $\bigcup_{x \in X_{n-1}} (\overline{g}^{-1}(x))^2 \overset{\text{closed}}{\subseteq} Y \times Y$, continuous and $\rho(Ta, Tb) = \rho(a, b)$ for $\overline{g}(a) = \overline{g}(b)$.

For $a, b \in X$ such that

$$\overline{g}(\pi'(a)) = \overline{g}(\pi'(b))$$

define

$$R(a, b) := \rho(\pi'(a), \pi'(b)).$$

- For any two out of $x, x', (x_k)$, R is defined.
- $R(x, x_k) = R(\tau^m x, \tau^m x_k)$ for all m.
- $F(x, x_k) \xrightarrow{k \to \infty} 0$, so there is a sequence (m_k) such that

$$d(\tau^{m_k} x, \tau^{m_k} x_k) \xrightarrow{k \to \infty} 0.$$

By continuity of ρ, we have that $R(x, x_k) = R(\tau^{m_k} x, \tau^{m_k} x_k) \xrightarrow{k \to \infty} 0$, and similarly $R(x', x_k) \to 0$. Hence $R(x, x') \xrightarrow{k \to \infty} 0$ by the triangle inequality. But x and x' don't depend on k, hence $R(x, x') = 0$. It follows that $\pi'(x) = \pi'(x')$ $\lightning$. $\square$

Theorem 4.41 (Beleznay-Foreman)**.** (1) For every $\eta < \omega_1$, there is a distal minimal flow of order η.

(2) The set of distal minimal flows is Π^1_1-complete.

(3) The order is a Π^1_1-rank. In particular {distal minimal flows of rank $<$ α} is Borel for all $\alpha < \omega_1$.

This was already stated as Theorem 4.14 in lecture 16 and should not have two numbers.

A few words on the proof: Let $\mathbb{K} = S^1$ and I a countable linear order. Let $\mathbb{K}^I$ be the product of $|I|$ many $\mathbb{K}$, $\mathbb{K}^{<i} := \mathbb{K}^{\{j : j < i\}}$ and $\pi_i \colon \mathbb{K}^I \to \mathbb{K}^{<i}$the projection.

Let $\mathbb{K}_I := \prod_{i \in I} C(\mathbb{K}^{<i}, \mathbb{K})$.

Fix some $(f_i)_{i \in I} \in \mathbb{K}_{\mathbb{I}}$. We build a flow acting on $\mathbb{K}$ from $(f_i)_{i \in I}$.

For this we define

$$\begin{aligned} E_I \colon \mathbb{K}_I &\longrightarrow C(\mathbb{K}^I, \mathbb{K}^I) \\ (f_i)_{i \in I} &\longmapsto \begin{pmatrix} \mathbb{K}^I & \longrightarrow & \mathbb{K}^I \\ x & \longmapsto & (f_i(\pi_i(x)) \cdot x_i)_{i \in I} \end{pmatrix} \end{aligned}$$

Example 4.42. Consider the following flow:

$$\begin{aligned} \tau \colon \mathbb{K}^3 &\longrightarrow \mathbb{K}^3 \\ (x, y, z) &\longmapsto (x \cdot \alpha, x^2 y, x y^3 z). \end{aligned}$$

Using

$$\begin{aligned} f_1 \colon \mathbb{K}^0 &\longrightarrow \mathbb{K} \\ x &\longmapsto \alpha, \end{aligned}$$

$$\begin{aligned} f_2 \colon \mathbb{K}^1 &\longrightarrow \mathbb{K} \\ x &\longmapsto x^2, \end{aligned}$$

$$\begin{aligned} f_3 \colon \mathbb{K}^2 &\longrightarrow \mathbb{K} \\ (x, y) &\longmapsto xy^3. \end{aligned}$$

we can write this as $\tau(x, y, z) = (x \cdot f_1 \circ \pi_1(x, y, z), y \cdot f_2\pi_2(x, y, z), z \cdot f_3\pi_3(x, y, z))$

Example 4.43. The skew shift can be written in this form as well. Consider $f_1 \colon x \mapsto \alpha$ and $f_n \colon (x_0, \ldots, x_{n-2}) \mapsto x_{n-2}$.

Theorem 4.44 (Beleznay Foreman)**.** Whenever $I = \eta$ for some $\eta < \omega_1$, then

$$\{\overline{f} \in \mathbb{K}_I : E_I(\overline{f}) \text{ is distal, minimal and of rank } \eta\}$$

is comeager in $\mathbb{K}_I$. In particular such flows exist.

Proof (sketch).

- Distality: For all $\overline{f} \in \mathbb{K}_I$, the flow $E_I\overline{f}$ is distal. This is the same as for iterated skew shifts.
- Minimality: [32]Let $\langle E_n : n < \omega\rangle$ be an enumeration of a countable basis for $\mathbb{K}^I$.

  For all n let
  $$U_n \coloneqq \{\overline{f} \in \mathbb{K}_I : \exists k \in \mathbb{Z}.\ f^k(\overline{1}) \in E_n\}$$
  where $f = E_I\overline{f}$ and $\overline{1} = (1, 1, 1, \ldots)$.

  Beleznay and Foreman showed that U_n is open and dense for all n.

  So if $\overline{f} \in \bigcap_n U_n$, then $\overline{1}$ is dense in $\overline{x} \mapsto f(\overline{x})$. Since the flow is distal, it suffices to show that one orbit is dense (cf. Theorem 4.21).
- The order of the flow is η: [33]Let $\overline{f} = (f_i)_{i\in I} \in \mathbb{K}_I$. Consider the flows we get from $(f_i)_{i<j}$ resp. $(f_i)_{i\leqslant j}$ denoted by $X_{<j}$ resp. $X_{\leqslant j}$. We aim to show that $X_{\leqslant j} \to X_{<j}$ is a maximal isometric extension for comeagerly many $\overline{f}$.

[32]Not relevant for the exam.
[33]Not relevant for the exam.

The following open dense sets are used to make sure that all isometric extensions are maximal and hence the order of the flow is η:

Fix a countable dense set $(\overline{x_n})$ in $\mathbb{K}^I$. For $\varepsilon \in \mathbb{Q}$ let

$$\begin{aligned}V_{j,m,n,\varepsilon} := \{\overline{f} \in \mathbb{K}_I : \\ &\text{if } \Pi_{j+1}(\overline{x}_n) = \Pi_{j+1}(\overline{x_m}),\\ &\text{then there are } k_m,\ k_n,\ \overline{z} \text{ such that}\\ &\pi_j(\overline{x_n}) = \pi_j(\overline{z}), \forall k > j+1.\ z_k = 1,\\ &d(f^{k_m}(\overline{x_m}), f^{k_m}(\overline{z})) < \varepsilon \text{ and}\\ &d(f^{k_n}(\overline{x_n}), f^{k_n}(\overline{z})) < \varepsilon\\ &\}\end{aligned}$$

Beleznay and Foreman show that this is open and dense.

□

[Lecture 23, 2024-01-19]

Notation 4.44.42. Let X be a Polish space and $\mathcal{P}$ a property of elements of X, then we say that $x_0 \in X$ is **generic** if

$$A_{\mathcal{P}} := \{x \in X : \mathcal{P}(x)\}$$

is comeager and $x_0 \in A_{\mathcal{P}}$.

For example let $X = \mathbb{K}_I$ and $\mathcal{P}$ the property of being a distal minimal flow.

Abuse of Notation 4.44.43. We will usually omit $\mathcal{P}$.

Let I be a linear order

Theorem 4.45 (Beleznay and Foreman)**.** The set of distal minimal flows is Π^1_1-complete.

Proof (sketch). [34]Consider $\mathrm{WO}(\mathbb{N}) \subseteq \mathrm{LO}(\mathbb{N})$. We know that this is Π^1_1-complete.

Let

$$\begin{aligned}S := \{x \in \mathrm{LO}(\mathbb{N}) : &x \text{ has a least element},\\ &\text{for any } t, \text{ there is } t \oplus 1, \text{ the successor of } t.\}\end{aligned}$$

S is Borel.[35]

[34]Not relevant for the exam.
[35]cf. Sheet 12, Exercise 1 (A.12.1)

We will construct a reduction

$$M \colon S \longrightarrow C(\mathbb{K}^{\mathbb{N}}, \mathbb{K})^{\mathbb{N}}.$$

We want that $\alpha \in \mathrm{WO}(\mathbb{N}) \iff M(\alpha)$ codes a distal minimal flow of rank α.

1. For any $\alpha \in S$, $M(\alpha)$ is a code for a flow which is coded by a generic $(f_i)_{i\in I}$. Specifically we will take a flow corresponding to some $(f_i)_{i\in I}$ which is in the intersection of all U_n, $V_{j,m,n,\frac{p}{q}}$ (cf. proof of Theorem 4.44).
2. If $\alpha \in \mathrm{WO}(\mathbb{N})$, then additionally $(f_i)_{i\in I}$ will code a distal minimal flow of ordertype α.

One can get a Borel map $S \ni \alpha \mapsto \{T_n^\alpha : n \in \mathbb{N}\}$, such that T_n^α is closed, $T_n^\alpha \neq \varnothing$, $\mathrm{diam}(T_n^\alpha) \xrightarrow{n\to\infty} 0$, $T_{n+1}^\alpha \subseteq T_n^\alpha$, $T_n^\alpha \subseteq W_n^\alpha$, where W_n^α is an enumeration of U_m^α,$V_{j,m,n,\frac{p}{q}}^\alpha$. Then $(f_i)_{i\in I} \in \bigcap_n T_n^\alpha$. $\square$

Lemma 4.46. Let $\{(X_\xi, T) : \xi \leqslant \eta\}$ be a normal quasi-isometric system and $\{(Y_i, T) : i \in I\}$ such that

(i) $I \in S$ and additionally I has a largest element.

(ii) Y_0 is the trivial flow and $Y_\infty = X_\eta$, where 0 and ∞ denote the minimal resp. maximal element of I.

(iii) $\forall i < j$

$$\begin{tikzcd} (X_\eta, T) \arrow[r, "\pi_j"] \arrow[dr, "\pi_i"'] & Y_j \arrow[d, "\pi_i^j"] \\ & Y_i \end{tikzcd}$$

(iv) If $i \in I$ is a limit (i.e. there does not exist an immediate predecessor), then (Y_i, T) is the inverse limit of $\{(Y_j, T) : j < i\}$ with respect to the factor maps.

(v) $(Y_{i\oplus 1}, T)$ is a maximal isometric extension of (Y_i, T) in (X_η, T).

Then I is well-ordered with $\mathrm{otp}(Y) = \eta + 1$.

Theorem 4.47 (Beleznay Foreman)**.** The order is a Π_1^1-rank.

Proof (sketch). [36] For the proof one shows that $\leqslant^*$ and $<^*$ are Π_1^1, where

(1) $p_1 \leqslant^* p_2$ iff p_1 codes a distal minimal flow and if p_2 also codes a distal minimal flow, then $order(p_1) \leqslant order(p_2)$.

[36] Not relevant for the exam.

(2) $p_1 <^* p_2$ iff p_1 codes a distal minimal flow and if p_2 also codes a distal minimal flow, then $order(p_1) < order(p_2)$.

One uses that (Y_{i+1}, T) is a maximal isometric extension of (Y_i, T) and (X, T) iff for all x_1, x_2 from a fixed countable dense set in X, for all i with $\pi_{i\oplus 1}(x_1) = \pi_{i\oplus 1}(x_2)$, there is a sequence (z_k) such that $\pi_i(z_k) = \pi_i(x_1)$, $F(z_k, x_1) \to 0$, $F(z_k, x_2) \to 0$. $\square$

Proposition 4.48. The order of a minimal distal flow on a separable, metric space is countable.

Proof. Let $(X, \mathbb{Z})$ be such a flow, i.e. X is separable, metric and compact.

Produce a normal quasi-isometric system

$$\{(X_\alpha, \mathbb{Z}) : \alpha \leqslant \beta\}$$

with $(X_\beta, \mathbb{Z}) = (X, \mathbb{Z})$. We need to show that $\beta < \omega_1$.

Let $\pi_\alpha \colon (X, \mathbb{Z}) \to (X_\alpha, \mathbb{Z})$. Fix $x_0 \in X$. For every α consider $\pi_\alpha^{-1}(\pi_\alpha(x_0)) = F_\alpha \overset{\text{closed}}{\subseteq} X$.

- For $\alpha_1 < \alpha_2 \leqslant \beta$ we have that $F_{\alpha_1} \supseteq F_{\alpha_2}$.
- For limits $\gamma \leqslant \beta$, we have that $F_\gamma = \bigcap_{\alpha < \gamma} F_\alpha$, since $(X_\gamma, \mathbb{Z})$ is the inverse limit of $\{(X_\alpha, \mathbb{Z}) : \alpha < \gamma\}$.
- For all α it is $F_{\alpha+1} \subsetneq F_\alpha$, because $\pi_\alpha^{\alpha+1} \colon (X_{\alpha+1}, \mathbb{Z}) \to (X_\alpha, \mathbb{Z})$ is not a bijection and all the fibers are isomorphic.

So $(F_\alpha)_{\alpha \leqslant \beta}$ is a strictly decreasing chain of closed subsets. But X is second countable, so β is countable: Let $\{U_n\} = \mathcal{B}$ be a countable basis and for α let $U_\alpha \in \mathcal{B}$ be such that $U_\alpha \cap F_\alpha = \varnothing$ and $U_\alpha \cap F_{\alpha+1} \neq \varnothing$. Then $\alpha \mapsto U_\alpha$ is an injection. $\square$

[Lecture 24, 2024-01-23]

4.4 Applications to Combinatorics

Definition 4.49. An **ultrafilter** on $\mathbb{N}$ (or any other set) is a family $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ such that

(1) $X \in \mathcal{U} \wedge X \subseteq Y \subseteq \mathbb{N} \implies Y \in \mathcal{U}$.

(2) $X, Y \in \mathcal{U} \implies X \cap Y \in \mathcal{U}$.

(3) $\varnothing \notin \mathcal{U}$, $\mathbb{N} \in \mathcal{U}$.

(4) For all $X \subseteq \mathbb{N}$, we have $X \in \mathcal{U} \vee \mathbb{N}\setminus X \in \mathcal{U}$.

Remark 4.49.44. • If $X \cup Y \in \mathcal{U}$ then $X \in \mathcal{U}$ or $Y \in \mathcal{U}$: Consider $((\mathbb{N}\backslash X) \cap (\mathbb{N}\backslash Y) = \mathbb{N}\backslash(X \cup Y)$.

- Every filter can be extended to an ultrafilter. (Zorn's lemma)

Definition 4.50. An ultrafilter is called **principal** or **trivial** iff it is of the form

$$\hat{n} = \{X \subseteq \mathbb{N} : n \in X\}.$$

Notation 4.50.45. Let $\varphi(\cdot)$ be a formula, where the argument is a natural number. Let $\mathcal{U}$ be an ultrafilter. We write

$$(\mathcal{U}n)\ \varphi(n)$$

for $\{n \in \mathbb{N} : \varphi(n)\} \in \mathcal{U}$. We say that $\varphi(n)$ holds for **$\mathcal{U}$-almost all** n.

Observe. *Let $\varphi(\cdot)$, $\psi(\cdot)$ be formulas.*

(1) $(\mathcal{U}n)\ (\varphi(n) \wedge \psi(m)) \iff (\mathcal{U}n)\varphi(n) \wedge (\mathcal{U}n)\psi(n)$.

(2) $(\mathcal{U}n)\ (\varphi(n) \vee \psi(m)) \iff (\mathcal{U}n)\ \varphi(n) \vee (\mathcal{U}n)\ \psi(n)$.

(3) $(\mathcal{U}n)\ \neg\varphi(n) \iff \neg(\mathcal{U}n)\ \varphi(n)$.

Lemma 4.51. Let X be a compact Hausdorff space. Let $\mathcal{U}$ be an ultrafilter. Then for every sequence (x_n) in X, there is a unique $x \in X$, such that

$$(\mathcal{U}n)\ (x_n \in G)$$

for every neighbourhood[a] G of x.

[a] $G \subseteq X$ is a neighbourhood iff $x \in \operatorname{int} G$.

Notation 4.51.46. In this case we write $x = \mathcal{U}\text{-}\lim\limits_{n} x_n$.

Proof of Lemma 4.51. [37] For metric spaces: Whenever we write $X = Y \cup Z$ we have $(\mathcal{U}n)x_n \in Y$ or $(\mathcal{U}n)x_n \in Z$.

So we can repeatedly chop the space in two pieces, one of them contains $\mathcal{U}$-almost all x_n, Then we restrict to this piece and continue.

For this to work, we need a finite collection $\mathcal{P}_n$ of closed sets for every n, such that $\bigcup \mathcal{P}_n = X$, $C \in \mathcal{P}_{n+1} \implies \exists C \subseteq D \in \mathcal{P}_n$ and $C_1 \supseteq C_2 \supseteq \ldots$, $C_i \in \mathcal{P}_i \implies |\bigcap_i C_i| = 1$. It is clear that we can do this for metric spaces.

[37] The proof from the lecture only works for metric spaces.

See Theorem 4.53 for the full proof. See Fact A.18.84 and Fact A.18.83 for a more general statement. $\square$

Let $\beta\mathbb{N}$ be the Čech-Stone compactification of $\mathbb{N}$, i.e. the set of all ultrafilters on $\mathbb{N}$ with the topology given by open sets $V_A = \{p \in \beta\mathbb{N} : A \in P\}$ for $A \subseteq \mathbb{N}$.

This is a compact Hausdorff space.[38] We can turn it into a compact semigroup: move facts
Consider $+\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. This gives an operation on principal ultrafilters (we identify $n \in \mathbb{N}$ with the corresponding principal filter). We want to extend this to all of $\beta\mathbb{N}$. Fix the first argument to get a function $\mathbb{N} \to \mathbb{N}, n \mapsto k + n$. For $\mathcal{U} \in \beta\mathbb{N}$ consider $\mathcal{U}\text{-}\lim_n (k+n)$. So for a fixed $k \in \mathbb{N}$ we get $k + \cdot\colon \beta\mathbb{N} \to \beta\mathbb{N}$, i.e. $+\colon \mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$. Fixing the second coordinate to be $\mathcal{V} \in \beta\mathbb{N}$, we get a function $+\mathcal{V}\colon \mathbb{N} \to \beta\mathbb{N}$. For $\mathcal{U} \in \beta\mathbb{N}$ consider $\mathcal{U}\text{-}\lim_n n + \mathcal{V}$. This gives $+\colon \beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$.

$$\mathcal{U} + \mathcal{V} = \{X \subseteq \mathbb{N} : \{m\colon \{n\colon m + n \in X\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

This is not commutative, but associative and $a \mapsto a + b$ is continuous for a fixed b, i.e. it is a left compact topological semigroup.

Let X be a compact Hausdorff space and let $T\colon X \to X$ be continuous.[39]

For any $\mathcal{U} \in \beta\mathbb{N}$, we define $T^{\mathcal{U}}$ by $T^{\mathcal{U}}(x) := \mathcal{U}\text{-}\lim_n T^n(x)$ for $x \in X$.

For fixed x, the map $\mathcal{U} \mapsto T^{\mathcal{U}}(x)$ is continuous.

(More generally, for every $f\colon \mathbb{N} \to X$ the extension $\tilde{f}\colon \beta\mathbb{N} \to X$ is continuous).

Note that for fixed $\mathcal{U}$, the map $x \mapsto T^{\mathcal{U}}(x)$ is not necessarily continuous.

Definition 4.52. Let X be a compact Hausdorff space and $T\colon X \to X$ continuous. A point $x \in X$ is **recurrent**, iff for every neighbourhood G of x, infinitely many n satisfy $T^n(x) \in G$.

A point $x \in X$ is **uniformly recurrent**, if for every neighbourhood G of x, there exists $M \in \mathbb{N}$, such that

$$\forall n.\ \exists k < M.\ T^{n+k}(x) \in G.$$

Fact 4.52.47. Let $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N}$ and $T\colon X \to X$ continuous for a compact Hausdorff space X. Then $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) = T^{\mathcal{U}+\mathcal{V}}(x)$.

[38] cf. Fact 4.52.49, Fact 4.52.50
[39] Note that this may not be a homeomorphism, i.e. we only get a $\mathbb{N}$-action but not a $\mathbb{Z}$-action.

Proof.

$$\begin{aligned}
T^{\mathcal{U}+\mathcal{V}}(x) &= (\mathcal{U}+\mathcal{V})\text{-}\lim_k T^k(x) \\
&= \mathcal{U}\text{-}\lim_m \mathcal{V}\text{-}\lim_n T^{m+n}(x) \\
&\overset{T^m \text{ continuous}}{=} \mathcal{U}\text{-}\lim_m T^m(\mathcal{V}\text{-}\lim_n T^n(x)) \\
&= T^{\mathcal{U}}(T^{\mathcal{V}}(x)).
\end{aligned}$$

□

Homework: Check the details that were omitted during the lecture.

[Lecture 25, 2024-01-26]

Let $\beta\mathbb{N}$ denote the set of ultrafilters on $\mathbb{N}$.

Fact 4.52.48.
- This is a topological space, where a basis consist of sets $V_A := \{p \in \beta\mathbb{N} : A \in p\}, A \subseteq \mathbb{N}$.

  (For $A, B \subseteq \mathbb{N}$ we have $V_{A\cap B} = V_A \cap V_B$ and $\beta\mathbb{N} = V_{\mathbb{N}}$.)
- Note also that for $A, B \subseteq \mathbb{N}$, $V_{A\cup B} = V_A \cup V_B$, $V_{A^c} = \beta\mathbb{N} \setminus V_A$.

Observe. *Note that the basis is clopen. In particular any closed set can be written as an intersection of sets of the form V_A:*

If F is closed, then $U = \beta\mathbb{N}\setminus F = \bigcup_{i\in I} V_{A_i}$, so $F = \bigcap_{i\in I} V_{\mathbb{N}\setminus A_i}$.

Fact 4.52.49. $\beta\mathbb{N}$ is Hausdorff.

Proof. Let $\mathcal{U} \neq \mathcal{V} \in \beta\mathbb{N}$. Then there is some $A \in \mathcal{U}\setminus\mathcal{V}$, so $A^c \in \mathcal{V}$, so $\mathcal{U} \in V_A$ and $\mathcal{V} \in V_A^c$. □

Fact 4.52.50. $\beta\mathbb{N}$ is compact.

Proof. Let $\{F_i\}_{i\in I}$ be non-empty and closed such that for any $i_1, \ldots, i_k \in I$, $k \in \mathbb{N}$, $\bigcap_{j=1}^k F_{i_j} \neq \emptyset$.

We need to show that $\bigcap_{i\in I} F_i \neq \emptyset$. Replacing each F_i by $V_{A^i_j}$ such that $F_i = \bigcap_{j\in J_i} V_{A^i_j}$ (cf.) we may assume that F_i is of the form V_{A_i}. We get $\{F_i = V_{A_i} : i \in I\}$ with the finite intersection property. Hence $\{A_i : i \in I\} =: \mathcal{F}_0$ has the finite intersection property.

Then $\mathcal{F} = \{A \subseteq \mathbb{N} : A \supseteq A_{i_1} \cap \ldots \cap A_{i_k}, k \in \mathbb{N}, i_1, \ldots, i_k \in I\}$ is a filter.

Let $\mathcal{U}$ be an ultrafilter extending $\mathcal{F}$. Then $\mathcal{U} \in \bigcap_{i\in I} V_{A_i} = \bigcap_{i\in I} F_i$. □

Fact 4.52.51. Consider $\mathbb{N}$ as a subspace of $\beta\mathbb{N}$ via $\mathbb{N} \hookrightarrow \beta\mathbb{N}, n \mapsto \hat{n} := \{A \subseteq \mathbb{N} : n \in A\}$. Then

- $\{\hat{n}\}$ is open in $\beta\mathbb{N}$ for all $n \in \mathbb{N}$.
- $\mathbb{N} \subseteq \beta\mathbb{N}$ is dense.

Theorem 4.53. For every compact Hausdorff space X, a sequence (x_n) in X, and $\mathcal{U} \in \beta\mathbb{N}$, we have that $\mathcal{U}\text{-}\lim_n x_n = x$ exists and is unique, i.e. for all $x \in G \overset{\text{open}}{\subseteq} X$ we have $\{n \in \mathbb{N} : x_n \in G\} \in \mathcal{U}$.

Proof. Towards a contradiction assume that there is no such x.

For every x take $x \in G_x \overset{\text{open}}{\subseteq} X$ such that $\{n \in \mathbb{N} : x_n \in G_x\} \notin \mathcal{U}$. So $\{G_x\}_{x \in X}$ is an open cover of X. Since X is compact, there exists a finite subcover $G_{x_1}, \ldots, G_{x_m}$.

But then

$$\begin{aligned}\mathbb{N} &= \{n \in \mathbb{N} : x_n \in \bigcup_{i=1}^m G_{x_i}\} \\ &= \underbrace{\bigcup_{i=1}^m \overbrace{\{n \in \mathbb{N} : x_n \in G_{x_i}\}}^{\notin \mathcal{U}}}_{\notin \mathcal{U}},\end{aligned}$$

since $B_1 \cup \ldots \cup B_m \in \mathcal{U} \iff \exists i < m.\ B_i \in \mathcal{U}$.

It is clear that $\mathcal{U}\text{-}\lim_n x_n$ is unique, since X is Hausdorff. □

Theorem 4.54. Let X be a compact Hausdorff space. For any $f : \mathbb{N} \to X$ there is a unique continuous extension $\tilde{f} : \beta\mathbb{N} \to X$.

Proof. Let

$$\begin{aligned}\tilde{f} : \beta\mathbb{N} &\longrightarrow X \\ \mathcal{U} &\longmapsto \mathcal{U}\text{-}\lim_n f(n).\end{aligned}$$

Exercise: Check that $\tilde{f}$ is continuous.

$\tilde{f}$ is uniquely determined, since $\mathbb{N} \subseteq \beta\mathbb{N}$ is dense. □

Trivial Nonsense† 4.54.52. β is a functor from the category of topological spaces to the category of compact Hausdorff spaces. It is left adjoint to the inclusion functor.

$\beta\mathbb{N}$ is equipped with $+$ which extends $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,

$$\mathcal{U} + \mathcal{V} = \{A \subseteq \mathbb{N} : (\mathcal{U}m)\,((\mathcal{U}n)\{m + n \in A\})\}.$$

This is associative, but not commutative.

Fact 4.54.53. $+ : \beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$ is left continuous, i.e. for $\mathcal{V}$ fixed, $\mathcal{U} \mapsto \mathcal{U} + \mathcal{V}$ is continuous.

Proof. Fix A and consider V_A. We need to show that the inverse image of V_A is open.

We have

$$\begin{aligned} \mathcal{U} + \mathcal{V} \in V_A &\iff A \in \mathcal{U} + \mathcal{V} \\ &\iff (\mathcal{U}_m)(\mathcal{V}_n)\{m + n \in A\} \\ &\iff \{m \in \mathbb{N} : (\mathcal{V}n)m + n \in A\} \in \mathcal{U} \\ &\iff \mathcal{U} \in V_{\{m \in \mathbb{N} : (\mathcal{V}n)m + n \in A\}}. \end{aligned}$$

□

Corollary 4.55. $(\beta\mathbb{N}, +)$ is a **compact semigroup**, i.e. it is compact, Hausdorff, associative and left-continuous.

So we can apply the Ellis-Numakura Lemma (4.19) to obtain

Corollary 4.56. There is $\mathcal{U} \in \beta\mathbb{N}$ such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

Observe. *Principal ultrafilters $\neq \hat{0}$ are not idempotent. We can restrict to $\beta\mathbb{N}\backslash\mathbb{N}$ to get an idempotent element that is not principal.*

Theorem 4.57 (Hindman)**.** If $\mathbb{N}$ is partitioned into finitely many sets, then there is is an infinite subset $H \subseteq \mathbb{N}$ such that all finite sums of distinct elements of H belong to the same set of the partition.

Proof of Theorem 4.57 (Galvin, Glazer). Let $\mathcal{U} \in \beta\mathbb{N}\backslash\mathbb{N}$ be such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$. Let P be the piece of the partition that is in $\mathcal{U}$. So $(\mathcal{U}n)n \in P$. Let us define a sequence $x_1, x_2, \ldots$

- $\mathcal{U}$ is idempotent, so $(\mathcal{U}n)(\mathcal{U}k)n + k \in P$. We get

$$(\mathcal{U}n)\,(n \in P \wedge (\mathcal{U}_k)n + k \in P)$$

. Pick x_1 that satisfies this, i.e. $x_1 \in P$ and $(\mathcal{U}_k)x_1 + k \in P$.

- $\mathcal{U}$ is idempotent, so
$$(\mathcal{U}n)[n \in P \wedge (\mathcal{U}_k)n + k \in P \wedge x_1 + n \in P \wedge (\mathcal{U}_k)x_1 + n + k \in P]$$
Take $x_2 > x_1$ that satisfies this.
- Suppose we have chosen $\langle x_i : i < n \rangle$. Since $\mathcal{U}$ is idempotent, we have
$$\begin{aligned}(\mathcal{U}n) \quad & n \in P \\ & \wedge (\mathcal{U}_k)n + k \in P \\ & \wedge \forall I \subseteq n.\ (\sum_{i \in I} x_i + n \in P) \\ & \wedge (\mathcal{U}_k)\left(\forall I \subseteq n.\ (\sum_{i \in I} x_i + n + k) \in P\right).\end{aligned}$$
Chose $x_n > x_{n-1}$ that satisfies this.

Set $H := \{x_i : i < \omega\}$. □

Next time we'll see another proof of this theorem.

[Lecture 26, 2024-01-30]

Let $T \colon X \to X$ be a continuous map. This gives $\mathbb{N} \curvearrowright X$.

Definition 4.58. A point $x \in X$ is called **uniformly recurrent** iff for each neighbourhood G of x, there is $M \in \mathbb{N}_+$, such that
$$\forall n \in \mathbb{N}.\ \exists k < m.\ T^{n+k}(x) \in G.$$

Definition 4.59. A pair $x, y \in X$ is **proximal**[a] iff for all neighbourhoods G of the diagonal[b] infinitely many n satisfy $(T^n(x), T^n(y)) \in G$.

[a] see also Definition 4.1, where we defined proximal for metric spaces

[b] recall that the diagonal is defined to be $\Delta := \{(x, x) : x \in X\}$

Theorem 4.60. Let X be a compact Hausdorff space and $T \colon X \to X$ continuous. Consider (X, T).Then for every $x \in X$ there is a uniformly recurrent $y \in X$ such that y is proximal to x.

We do a second proof of Theorem 4.57:

Proof of Theorem 4.57 (Furstenberg). A partition of $\mathbb{N}$ into k-many pieces can be viewed as a function $f \colon \mathbb{N} \to k$.

Let $X = k^{\mathbb{N}}$ be the set of all such functions. Equip X with the product topology. Then X is compact and Hausdorff.

Let $T: X \to X$ be the shift given by

$$\begin{aligned} T: k^{\mathbb{N}} &\longrightarrow k^{\mathbb{N}} \\ (y: \mathbb{N} \to k) &\longmapsto \begin{pmatrix} \mathbb{N} & \longrightarrow & k \\ n & \longmapsto & y(n+1), \end{pmatrix} \end{aligned}$$

i.e. $T(y)(n) = y(n+1)$.

Let x be the given partition. We want to find an infinite set H for x as in the theorem. Let y be uniformly recurrent and proximal to x.

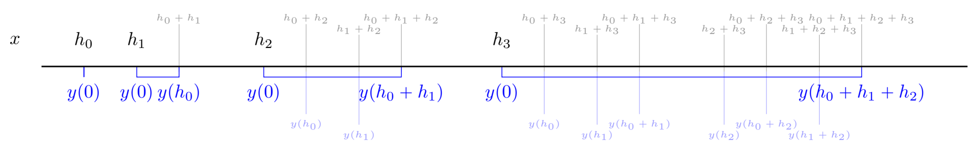
- Since x and y are proximal, we get that for every $N \in \mathbb{N}$, there are infinitely many n such that $T^n(x)|_N = T^n(y)|_N$.[40]
- Consider the neighbourhood

  $$G_n := \{z \in X : z|_n = y|_n\}$$

  of y. By the uniform recurrence of y, we get that[41]

  $$\begin{aligned} &\forall n.\ \exists N.\ \forall r.\ (y(r), y(r+1), \ldots, y(r+N-1)) \\ &\qquad\qquad \text{contains } (y(0), y(1), \ldots, y(n)) \text{ as a subsequence.} \end{aligned}$$

Consider $y(0)$. We will prove that this color works and construct a corresponding H.



- Let $G_0 := [y(0)]$ and let N_0 be such that

  $$\forall r.\ (y(r), \ldots, y(r + N_0 - 1)) \text{ contains } y(0).$$

  By proximality, there exist infinitely many r such that $(y(r), \ldots, (y(r + N_0 - 1)) = (x(r), \ldots, x(r + N_0 - 1))$. Fix $h_0 \in \mathbb{N}$ such that $x(h_0) = y(0)$.
- Let $G_{n_0} = [(y(0), \ldots, y(h_0)]$. Choose N_1.

  For all r, $(y(r), \ldots, y(r+N-1))$ contains $(\underbrace{y(0)}_{=C}, \ldots, \underbrace{y(h_0)}_{=C})$.

  Pick $r > h_0$ such that $(x(r), \ldots, x(r+N-1))$ contains $(y(0), \ldots y(h_0))$. Let $(x(r+s), \ldots, x(r+s+h_0)) = (y(0), \ldots, y(h_0))$. Then set $h_1 = r + s$.

  Then $x(h_0) = c$, $x(h_1) = y(0) = c$ and $x(h_0 + h_1) = y(h_0) = c$.

[40] Consider $G_N = \{(a, b) \in X^2 : a|_N = b|_N\}$ This is a neighbourhood of the diagonal.
[41] Note that here we might need to choose a bigger N than the M in Definition 4.58, but $2M$ suffices.

- Let $G_{h_0+h_1} = [y(0), \dots, y(h_0+h_1)]$. Let $r > h_0 + h_1$. Choose N_2 large enough such that $(y(0), \dots, y(h_0+h_1))$ is contained in $(x(r), \dots, x(r+N-1))$. Let $(y(0), \dots, y(h_0+h_1)) = (x(r+s), \dots, x(r+s+N-1))$.
- Repeat this: Inductively choose h_i such that $x(s+h_i) = y(s+h_i) = c$ for all sums s of subsets of $\{h_0, \dots, h_{i-1}\}$. To do this, find N_i such that every N_i consecutive terms of y contain $(y(0), \dots, y(\sum_{j<i} h_j))$. Then find $h_i > h_{i-1}$ such that $(x(h_i), \dots, x(\sum_{j<i} h_j)) = (y(0), \dots, y(\sum_{j<i} h_j))$.

$\square$

In order to prove Theorem 4.60, we need to rephrase the problem in terms of $\beta\mathbb{N}$:

Theorem 4.61. Let X be a compact Hausdorff space. Let $T\colon X \to X$ be continuous.

(1) $x \in X$ is recurrent iff $T^{\mathcal{U}}(x) = x$ for some $\mathcal{U} \in \beta\mathbb{N}\backslash\mathbb{N}$.

(2) $x \in X$ is uniformly recurrent iff for every $\mathcal{V} \in \beta\mathbb{N}$, there is $\mathcal{U} \in \beta\mathbb{N}$ with $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) = x$.

(3) $x, y \in X$ are proximal iff there is $\mathcal{U} \in \beta\mathbb{N}$ such that $T^{\mathcal{U}}(x) = T^{\mathcal{U}}(y)$.

Proof of Theorem 4.61 (sketch). We only prove (2) here, as it is the most interesting point.

other parts will be in the official notes

Subproof ((2), $\implies$). Suppose that x is uniformly recurrent. Take some $\mathcal{V} \in \beta\mathbb{N}$. Let G_0 be a neighbourhood of x. Then $x \in G \subseteq G_0$, where G is a closed neighbourhood, i.e. $X \in \operatorname{int} G$.

Let M be such that
$$\forall n.\ \exists k < M.\ T^{n+k}(x) \in G.$$
So there is a $k < M$ such that
$$(\mathcal{V}n)T^{n+k}(x) \in G.$$
Hence
$$(\mathcal{V}n)T^n(x) \in \underbrace{T^{-k}(\overbrace{\ G\ }^{\text{closed}})}_{\text{closed}}.$$
Therefore $\mathcal{V}\text{-}\lim_n T^n(x) \in T^{-k}(G)$. So $T^k(T^{\mathcal{V}}(x)) \in G \subseteq G_0$.

We have shown that for every open neighbourhood G of x, the set $Y_G = \{k \in \mathbb{N} : T^k(T^{\mathcal{V}}(x)) \in G\} \neq \varnothing$. The sets $\{Y_G : G \text{ open neighbourhood of } G\}$ form a filter basis,[42] since $Y_{G_1} \cap Y_{G_2} = Y_{G_1 \cap G_2}$. Let $\mathcal{U}$ be an ultrafilter containing all the Y_G.

Then
$$(\mathcal{U}k) R^k(T^{\mathcal{V}})(x)) \in G$$
i.e. $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) \in \overline{G}$.

Since we get this for every neighbourhood, it follows that $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) = x$. ■

[Lecture 27, 2024-02-02]

Continuation of proof of Theorem 4.61. Subproof ((2), $\Longleftarrow$, sketch). Assume that x is not uniformly recurrent. Then there is a neighbourhood $G \ni x$ such that for all $M \in \mathbb{N}$
$$Y_M = \{n \in \mathbb{N} : \forall k < M.\ T^{n+k}(x) \notin G\} \neq \varnothing.$$
Note that $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \ldots$ Take $\mathcal{V} \in \beta\mathbb{N}$ containing all Y_n.

We aim to show that there is no $\mathcal{U} \in \beta\mathbb{N}$ such that $T_{\mathcal{U}}(T_{\mathcal{V}}(x)) = x$. Towards a contradiction suppose that such $\mathcal{U}$ exists.

For every $k+1$ we have $Y_{k+1} \in \mathcal{V}$. In particular
$$\{n \in \mathbb{N} : T^{n+k}(x) \notin G\} \supseteq Y_{k+1},$$
so
$$(\mathcal{V}n) T^{n+k}(x) \notin G,$$
i.e.
$$(\mathcal{V}n) T^n(x) \notin \underbrace{T^{-k}(G)}_{\text{open}}.$$
Thus
$$\underbrace{\mathcal{V}\text{-}\lim_n T^n(x)}_{T^{\mathcal{V}}(x)} \notin T^{-k}(G).$$
We get that
$$\forall k.\ T^k(T^{\mathcal{V}}(x)) \notin G.$$
It follows that $\forall \mathcal{U} \in \beta\mathbb{N}.\ T^{\mathcal{U}}(T^{\mathcal{V}}(x)) \notin G$.

■

□

Take $X = \beta\mathbb{N}$, $S \colon \beta\mathbb{N} \to \beta\mathbb{N}$, $S(\mathcal{U}) = \hat{1} + \mathcal{U}$. Then
$$S^{\mathcal{V}}(\mathcal{U}) = \mathcal{V}\text{-}\lim_n S^n(\mathcal{U}) = \mathcal{V}\text{-}\lim_n (\hat{n} + \mathcal{U}) = \mathcal{V}\text{-}\lim_n \hat{n} + \mathcal{U} = \mathcal{V} + \mathcal{U}.$$

[42] The sets and their supersets form a filter.

Corollary 4.62. $\mathcal{U}$ is recurrent iff

$$\exists \mathcal{V} \in \beta\mathbb{N}\backslash\mathbb{N}.\ S^{\mathcal{V}}(\mathcal{U}) = \mathcal{U}.$$

$\mathcal{U}$ is uniformly recurrent iff

$$\forall \mathcal{V}.\ \exists \mathcal{W}.\ \mathcal{W} + \mathcal{V} + \mathcal{U} = \mathcal{U}.$$

$\mathcal{U}_1$ and $\mathcal{U}_2$ are proximal iff $\exists \mathcal{V}.\ \mathcal{V} + \mathcal{U}_1 = \mathcal{V} + \mathcal{U}_2$.

Definition 4.63. We say that $I \subseteq \beta\mathbb{N}$ is a **left ideal** , if

$$\forall \mathcal{U} \in I.\ \forall \mathcal{V} \in \beta\mathbb{N}.\ \mathcal{V} + \mathcal{U} \in I.$$

Theorem 4.64. (1) $\mathcal{U}$ is uniformly recurrent in $\beta\mathbb{N}$ iff $\mathcal{U}$ belongs to a minimal[a] (closed) left ideal in $\beta\mathbb{N}$.

(2) $\mathcal{U}$ is an idempotent in $\beta\mathbb{N}$ iff $\mathcal{U}$ belong to a minimal closed subsemigroup of $\beta\mathbb{N}$.

[a]wrt. $\subseteq$

Proof. (1) Note that any $\mathcal{U} \in \beta\mathbb{N}$ yields a left ideal $\beta\mathbb{N} + \mathcal{U}$. It is closed, since it is the image of $\beta\mathbb{N}$ under the continuous maps $\mathcal{V} \mapsto \mathcal{V} + \mathcal{U}$ and $\beta\mathbb{N}$ is compact. $\mathcal{U}$ belongs to a minimal left ideal iff $\beta\mathbb{N} + \mathcal{U}$ is minimal, since every ideal containing $\mathcal{U}$ contains $\beta\mathbb{N}+\mathcal{U}$. Note that $\beta\mathbb{N}+\mathcal{V}+\mathcal{U} \subseteq \beta\mathbb{N}+\mathcal{U}$ and if $I \subsetneq \beta\mathbb{N} + \mathcal{U}$, we have $\mathcal{V}_0 = \mathcal{V} + \mathcal{U} \in I$ and $\beta\mathbb{N} + \mathcal{V} + \mathcal{U} \subseteq \beta\mathbb{N} + \mathcal{U}$. So $\mathcal{U}$ belongs to a minimal left ideal iff

$$\forall \mathcal{V} \in \beta\mathbb{N}.\ \beta\mathbb{N} + \mathcal{V} + \mathcal{U} = \beta\mathbb{N} + \mathcal{U}.$$

This is the case iff

$$\underbrace{\forall \mathcal{V}.\ \exists \mathcal{W}.\ \mathcal{W} + \mathcal{V} + \mathcal{U} = \mathcal{U}.}_{\mathcal{V} \text{ uniformly recurrent}}$$

(For one direction take $\mathcal{W}$ such that $\mathcal{W} + \mathcal{V} + \mathcal{U} = \hat{0} + \mathcal{U}$. For the other direction note that for every $\mathcal{V}_0$, $\mathcal{V}_0 + \mathcal{U}$ can be written as $\mathcal{V}_0 + \mathcal{W} + (\mathcal{V} + \mathcal{U})$. Where we take $\mathcal{W}$ such that $\mathcal{W} + \mathcal{V} + \mathcal{U} = \mathcal{U}$.

(2) This is very similar to the proof of the Ellis-Numakura Lemma (4.19).

If $\mathcal{U}$ is idempotent, then $\{\mathcal{U}\}$ is a semigroup. Let C be a minimal closed subsemigroup of $\beta\mathbb{N}$. Then $C + \mathcal{U}$ is a closed subsemigroup. By minimality, we get $C = C + \mathcal{U}$.

Let $D = \{\mathcal{V} \in C.\ \mathcal{V} + \mathcal{U} = \mathcal{U}\}$. We have $D \neq \varnothing$. D is a closed semigroup, so $D = C$ be minimality. Hence $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

□

Corollary 4.65. Idempotent and uniformly recurrent elements exist.

Proof. Use Theorem 4.64 and Zorn's lemma. □

Theorem 4.66. (1) $\implies$ (2) $\implies$ (3) where

(1) $\mathcal{U}$ is uniformly recurrent and proximal to $\hat{0}$.

(2) $\mathcal{U}$ is an idempotent.

(3) $\mathcal{U}$ is recurrent and proximal to $\hat{0}$.

Proof. (1) $\implies$ (2): Let $\mathcal{U}$ be uniformly recurrent and proximal to $\hat{0}$. Take $\mathcal{V}$ such that $\mathcal{V} + \mathcal{U} = \mathcal{V} + \hat{0} = \mathcal{V}$.

Since $\mathcal{U}$ is uniformly recurrent, there exists $\mathcal{W}$ such that $\mathcal{W} + \mathcal{V} + \mathcal{U} = \mathcal{U}$, i.e. $\mathcal{W} + \mathcal{V} = \mathcal{U}$. Then $\mathcal{U} + \mathcal{U} = \mathcal{W} + \mathcal{V} + \mathcal{U} = \mathcal{U}$.

(2) $\implies$ (3): Let $\mathcal{U}$ be an idempotent. Then $\mathcal{V} + \mathcal{U} = \mathcal{V}$ (proximal to 0) and $\mathcal{V} + \mathcal{U} = \mathcal{U}$ (recurrent) are satisfied for $\mathcal{V} := \mathcal{U}$. □

Corollary 4.67. $\mathcal{U}$ is uniformly recurrent and proximal to 0 iff $\mathcal{U}$ is an idempotent and belongs to some minimal left ideal of $\beta\mathbb{N}$.

Finally:

Proof of Theorem 4.60. Let $T\colon X \to X$ and $x \in X$. We want to find $y \in X$ such that y is uniformly recurrent and proximal to x.

We first prove a version for ultrafilters and then transfer it to X.

There exists a uniformly recurrent $\mathcal{V} \in \beta\mathbb{N}$. So for any $\mathcal{W}$, $\mathcal{W}+\mathcal{V}$ is also uniformly recurrent: Take $\mathcal{V}_0$. We need to find $\mathcal{X}$ such that $\mathcal{X} + \mathcal{V}_0 + \mathcal{W} + \mathcal{V} = \mathcal{W} + \mathcal{V}$. By uniform recurrence of $\mathcal{V}$ we find $\mathcal{X}'$ such that $\mathcal{X}' + (\mathcal{V}_0 + \mathcal{W}) + \mathcal{V} = \mathcal{V}$. Then $\mathcal{X} = \mathcal{W} + \mathcal{X}'$ works. So all elements of $\beta\mathbb{N} + \mathcal{V}$ are uniformly recurrent. It is a closed ideal and hence a closed semigroup. So $\beta\mathbb{N} + \mathcal{V}$ contains a minimal closed semigroup. In particular, there exists an idempotent $\mathcal{U} \in \beta\mathbb{N} + \mathcal{V}$.

$\mathcal{U}$ is idempotent and uniformly recurrent hence it is proximal to 0.

Now let us consider X. Take $y = T^{\mathcal{U}}(x)$.

Claim 4.60.1. *y uniformly recurrent.*

Subproof. Recall that $T^{\mathcal{V}_1+\mathcal{V}_2} = T^{\mathcal{V}_1} \circ T^{\mathcal{V}_2}$.

Since $\mathcal{U}$ is uniformly recurrent, $\forall \mathcal{V}.\ \exists \mathcal{W}.\ \mathcal{W} + \mathcal{V} + \mathcal{U} = \mathcal{U}$, i.e. $T^{\mathcal{W}+\mathcal{V}+\mathcal{U}}(x) = T^{\mathcal{W}}(T^{\mathcal{V}}(y)) = T^{\mathcal{U}}(x) = y$. ■

Claim 4.60.2. *y is proximal to x.*

Subproof. $\mathcal{U}$ is proximal to 0. So $\exists \mathcal{V}.\ \mathcal{V}+\mathcal{U} = \mathcal{V}+\hat{0} = \mathcal{V}$, i.e. $T^{\mathcal{V}}(y) = T^{\mathcal{V}+\mathcal{U}}(x) = T^{\mathcal{V}}(x)$. Thus x and y are proximal. ■

□

A Tutorial and Exercises

[Tutorial 01, 2023-10-17]

Fact A.0.54. A countable product of separable spaces $(X_n)_{n\in\mathbb{N}}$ is separable.

Proof. Choose a countable dense subset $D_n \subseteq X_n$ Fix some point $(a_1, a_2, \ldots) \in \prod_n X_n$ and consider $\bigcup_{i\in\mathbb{N}} \prod_{n\leqslant i} D_n \times \prod_{n>i}\{a_n\}$. □

Fact A.0.55.
- Let X be a topological space. Then X 2^{nd} countable $\implies$ X separable.
- If X is a metric space and separable, then X is 2^{nd} countable.

Proof. For the first point, choose some point from every basic open set.

For the second point consider balls of rational radius around the points of a countable dense subset. □

Definition A.1. A topological space is **Lindelöf** iff every open cover has a countable subcover.

Fact A.1.56. Let X be a metric space. If X is Lindelöf, then it is 2^{nd} countable.

Proof. For all $q \in \mathbb{Q}$ consider the cover $B_q(x), x \in X$ and choose a countable subcover. The union of these subcovers is a countable base. □

Fact A.1.57. Let X be a topological space. If X is 2^{nd} countable, then it is Lindelöff.

Proof. Let $A_0, A_1, \ldots$ be a countable base.

Let $\{U_i\}_{i\in I}$ be a cover. Consider $J := \{j : \exists i \in I.\ A_j \in U_i\}$. For every $j \in J$ choose a U_i such that $A_j \subseteq U_j$. Let $I' \subseteq I$ be the subset of chosen indices. Then $\{U_i\}_{i\in I'}$ is a countable subcover. □

Remark A.1.58. For metric spaces the notions of being 2^{nd} countable, separable and Lindelöf coincide.

In arbitrary topological spaces, Lindelöf is the weakest of these notions.

Definition† A.1.59. A metric space X is **totally bounded** iff for every $\varepsilon > 0$ there exists a finite set of points $x_1, \ldots, x_n$ such that $X = \bigcup_{i=1}^{n} B_\varepsilon(x_i)$.

A.1 Sheet 1

[Tutorial 02, 2023-10-24]

A.1.1 Exercise 1

Let (X, d) be a metric space and $\varnothing \neq A \subseteq X$. Let $d(x, A) := \inf(d(x, a) : a \in A\}$.

- $d(-, A)$ is uniformly continuous:

  Clearly $|d(x, A) - d(y, A)| \leqslant d(x, y)$. Add details

- $d(x, A) = 0 \iff x \in \overline{A}$.

  $d(x, A) = 0$ iff there is a sequence in A converging towards x iff $x \in \overline{A}$.

A.1.2 Exercise 2

Let X be a discrete space. For $f, g \in X^{\mathbb{N}}$ define

$$d(f, g) := \begin{cases} (1 + \min\{n : f(n) \neq g(n)\})^{-1} & : f \neq g, \\ 0 & : f = g. \end{cases}$$

(a) d is an **ultrametric**, i.e. $d(f, g) \leqslant \max\{d(f, h), d(g, h)\}$ for all $f, g, h \in X^{\mathbb{N}}$:

Let $f, g, h \in X^{\mathbb{N}}$.

We need to show that $d(f, g) \leqslant \max(d(f, h), d(g, h))$.

If $f = g$ this is trivial. Otherwise let n be minimal such that $f(n) \neq g(n)$. Then $h(n) \neq f(n)$ or $h(n) \neq g(n)$ must be the case. W.l.o.g. $h(n) \neq f(n)$. Then $d(f, g) = \frac{1}{1+n} \leqslant d(f, h)$.

(b) d induces the product topology on $X^{\mathbb{N}}$:

It suffices to show that the ε-balls with respect to d are exactly the basic open set of the product topology, i.e. the sets of the form

$$\{x_1\} \times \ldots \times \{x_n\} \times X^{\mathbb{N}}$$

for some $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$.

Let $\varepsilon > 0$. Let n be minimal such that $\frac{1}{1+n} \geqslant \varepsilon$. Then $B_\varepsilon((x_i)_{i\in\mathbb{N}}) = \{x_1\} \times \{x_n\} \times X^{\mathbb{N}}$. Since $\mathbb{N} \ni n \mapsto \frac{1}{1+n}$ is injective, every basic open set of the product topology can be written in this way.

(c) d is complete:

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to d. For $n \in \mathbb{N}$ take $N_n \in \mathbb{N}$ such that $d(f_i, f_j) < \frac{1}{1+n}$. Clearly $f_i(n) = f_j(n)$ for all $n > N_n$.

Define $f \in X^{\mathbb{N}}$ by $f(n) := f_{N_n}(n)$. Then $(f_n)_{n\in\mathbb{N}}$ converges to f, since for all $n > N_n$ f_n

(d) If X is countable, then $X^{\mathbb{N}}$ with the product topology is a Polish space:

(We assume that X is non-empty, as otherwise the claim is wrong)

We need to show that there exists a countable dense subset. To this end, pick some $x_0 \in X$ and consider the set $D := \bigcup_{n\in\mathbb{N}}(X^n \times \{x_0\}^{\mathbb{N}})$. Since X is countable, so is D. Take some $(a_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and consider $B := B_\varepsilon((a_n)_{n\in\mathbb{N}})$. Let m be such that $\frac{1}{1+m} < \varepsilon$. Then $(b_n)_{n\in\mathbb{N}} \in B \cap D$, where $b_n := a_n$ for $n \leqslant m$ and $b_n := x_0$ otherwise. Hence D is dense.

A.1.3 Exercise 3

Consider $\mathbb{N}$ as a discrete space and $\mathbb{N}^{\mathbb{N}}$ with the product topology. Let

$$S_\infty = \{f\colon \mathbb{N} \to \mathbb{N} \text{ bijective}\} \subseteq \mathbb{N}^{\mathbb{N}}.$$

(a) S_∞ is a Polish space:

From Sheet 1, Exercise 2 (A.1.2) we know that $\mathbb{N}^{\mathbb{N}}$ is Polish. Hence it suffices to show that S_∞ is G_δ with respect to $\mathbb{N}^{\mathbb{N}}$.

Consider the sets $I := \bigcap_{(i,j)\in\mathbb{N}^2, i<j}\{f \in \mathbb{N}^{\mathbb{N}} | f(i) \neq f(j)\}$ and $S := \bigcap_{n\in\mathbb{N}}\{f \in \mathbb{N}^{\mathbb{N}} | n \in \operatorname{im} f\}$.

We have that $\{f \in \mathbb{N}^{\mathbb{N}} | f(i) \neq f(j)\} = \bigcup_{n\in\mathbb{N}} \mathbb{N}^{i-1} \times \{n\} \times \mathbb{N}^{i-j-1} \times (\mathbb{N}\backslash\{n\}) \times \mathbb{N}^{\mathbb{N}}$ is open. Hence I is G_δ.

Furthermore $\{f \in \mathbb{N}^{\mathbb{N}} | n \in \operatorname{im} f\} = \bigcup_{k\in\mathbb{N}} \mathbb{N}^k \times \{n\} \times \mathbb{N}^{\mathbb{N}}$j is open, thus S is G_δ as well. In particular $S \cap G$ is G_δ. Since I is the subset of injective functions and S is the subset of surjective functions, we have that $S_\infty = I \cap S$.

(b) S_∞ is not locally compact:

Consider the point $x = (i)_{i\in\mathbb{N}} \in S_\infty$. Let $x \in B$ be open. We need to show that there is no closed compact set $C \supseteq B$ W.l.o.g. let $B = (\{0\} \times \ldots \times \{n\} \times \mathbb{N}^{\mathbb{N}}) \cap S_\infty$ for some $n \in \mathbb{N}$. Let $C \supseteq B$ be some closed set. Consider the open covering

$$\{S_\infty \backslash B\} \cup \{B_j | j > n\}.$$

where

$$B_j := (\{0\} \times \ldots \times \{n\} \times \{j\} \times \mathbb{N}^{\mathbb{N}}) \cap S_\infty.$$

Clearly there cannot exist a finite subcover as B is the disjoint union of the B_j.

A.1.4 Exercise 4

Fact A.1.60. Let X be a compact Hausdorff space. Then the following are equivalent:

(i) X is Polish,

(ii) X is metrisable,

(iii) X is second countable.

Proof. (i) $\implies$ (ii) clear

(i) $\implies$ (iii) clear

(ii) $\implies$ (i) Consider the cover $\{B_\varepsilon(x) | x \in X\}$ for every $\varepsilon \in \mathbb{Q}$ and chose a finite subcover. Then the midpoints of the balls from the cover form a countable dense subset.

The metric is complete as X is compact. (For metric spaces: compact $\iff$ seq. compact $\iff$ complete and totally bounded)

(iii) $\implies$ (ii) Use Urysohn's metrisation theorem and the fact that compact Hausdorff spaces are normal □

Let X be compact Polish[43] and Y Polish. Let $\mathcal{C}(X,Y)$ be the set of continuous functions $X \to Y$. Consider the **uniform metric** $d_u(f,g) := \sup_{x\in X} |d(f(x), g(x))|$. Clearly d_u is a metric.

Claim 1. *d_u is complete.*

Subproof. Let (f_n) be a Cauchy sequence in $\mathcal{C}(X,Y)$. A Y is complete, there exists a pointwise limit f.

f_n converges uniformly to f:

$$d(f_n(x), f(x)) \leqslant \overbrace{d(f_n(x), f_m(x))}^{(f_n) \text{ is Cauchy}} + \underbrace{d(f_m(x), f(x))}_{\text{small for appropriate } m}.$$

f is continuous by the uniform convergence theorem. ■

Claim 2. *There exists a countable dense subset.*

Subproof. Fix a metric d_X on X defining its topology. Let

$$C_{m,n} := \{f \in \mathcal{C}(X,Y) : \forall x, y \in X. \left(d_X(x,y) < \frac{1}{m+1} \implies d(f(x), f(y)) < \frac{1}{n+1}\right)\}.$$

[43] compact metrisable $\implies$ compact Polish

Choose $X_m \subseteq X$ finite with $X \subseteq \bigcup_{x \in X_m} B_{\frac{1}{m+1}}(x)$. Let $D_{m,n} \subseteq C_{m,n}$ be countable, such that for every $f \in C_{m,n}$ and every $\eta > 0$, there is $g \in D_{m,n}$ with $d(f(y), g(y)) < \frac{\eta}{3}$ for each $y \in X_m$. Then $\bigcup_{m,n} D_{m,n}$ is dense in $\mathcal{C}(X,Y)$: Indeed if $f \in \mathcal{C}(X,Y)$ and $\eta > 0$, we find $n > \frac{3}{\eta}$ and m such that $f \in C_{m,n}$, since f is uniformly continuous. Let $g \in D_{m,n}$ be such that $\forall y \in X_m.\ d(f(y), g(y)) < \frac{1}{n+1}$. We have $d_u(f,g) \leqslant \eta$, since for every $x \in X$, we find $y \in X_m$ with $d_X(x,y) < \frac{1}{m+1}$, hence

$$\begin{aligned} d_Y(f(x), g(x)) &\leqslant d_Y(f(x), f(y)) + d_Y(f(y), g(y)) + d_Y(g(y), g(x)) \\ &\leqslant \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} \leqslant \eta. \end{aligned}$$

■

A.2 Sheet 2

[Tutorial 03, 2023-10-31]

Remark A.1.61. F_σ stands for **fermé sum denumerable**.

A.2.1 Exercise 1

Let X be a Polish space. Then there exists an injection $f\colon X \to 2^\omega$ such that for each $n < \omega$, the set $f^{-1}(\{(y_n) \in 2^\omega : y_n = 1\})$ is open. Moreover if $V \subseteq 2^\omega$ is closed, then $f^{-1}(V)$ is G_δ.

Let $(U_i)_{i<\omega}$ be a countable base of X. Define

$$\begin{aligned} f\colon X &\longrightarrow 2^\omega \\ x &\longmapsto (x_i)_{i<\omega} \end{aligned}$$

where $x_i = 1$ iff $x \in U_i$ and $x_i = 0$ otherwise. Then $f^{-1}(\{y = (y_n) \in 2^\omega | y_n = 1\}) = U_n$ is open. We have that f is injective since X is T1.

Let $f\colon X \hookrightarrow 2^\omega$ be such that $f^{-1}(\{y = (y_n) \in 2^\omega | y_n = 1\})$.

Let $V \subseteq 2^\omega$ be closed. Then $2^\omega \setminus V$ is open, i.e. has the form $\bigcup_{i \in I}((\prod_{j<n_j} X_{i,j}) \times 2^\omega)$ for some $X_{i,j} \subseteq 2$. As 2^ω is second countable, we may assume I to be countable.

Then $V = \bigcap_{i \in I}\left(2^\omega \setminus ((\prod_{i<n_j} X_{i,j}) \times 2^\omega)\right)$. Since f is injective, we have $f^{-1}(\bigcap_{a \in A} a) = \bigcap_{a \in A} f^{-1}(a)$. Thus it suffices to show that $f^{-1}(2^\omega \setminus ((\prod_{i<n} X_i) \times 2^\omega))$ is G_δ, as a countable intersection of G_δ-sets is G_δ.

We have that $U_k := f^{-1}(\{y = (y_i) \in 2^\omega : y_k = 1\})$ is open. Since f is injective $f^{-1}(\{y = (y_i) \in 2^\omega : y_k = 0\}) = X \setminus U_k$ is closed, in particular it is G_δ. Let $x = (x_1, \ldots, x_n) \in 2^n \setminus (\prod_{i<n} X_i)$. Then $f^{-1}(\{x\} \times 2^\omega) = \bigcap_{i<n} \bigcap U_n'$ is G_δ, were $U_i' = U_i$ if $x_k = i$ and $U_i' = X \setminus U_i$ otherwise.

Since $2^n \setminus (\prod_{i<n} X_i)$ is finite, we get that $f^{-1}(2^\omega \setminus ((\prod_{i<n} X_i) \times 2^\omega))$ is G_δ as a finite union of G_δ sets.

A.2.2 Exercise 2

Let X be a Polish space. Then X is homeomorphic to a closed subspace of $\mathbb{R}^\omega$:

handwritten solution

A.2.3 Exercise 3

Example A.2. Consider

$$\begin{aligned} f\colon \mathbb{R} &\longrightarrow [0,1] \\ \frac{p}{q} &\longmapsto \frac{1}{q} \\ \mathbb{R}\backslash\mathbb{Q} \ni x &\longmapsto 0 \end{aligned}$$

Then $\operatorname{osc}_f(\frac{p}{q}) = \frac{1}{q}$ and $\operatorname{osc}_f(x) = 0$ for $x \notin \mathbb{Q}$.

Definition A.3. We say that $f\colon X \to Y$ is continuous at $a \in X$, if for N a neighbourhood of $f(a)$ (i.e. there exists $f(a) \in U \overset{\text{open}}{\subseteq} N$, then $f^{-1}(N)$ is a neighbourhood of a.

Theorem A.4 (Kuratowski)**.** Let X be metrizable, Y completely metrizable, $S \subseteq X$ and $f\colon S \to Y$ continuous. Then f can be extended to a continuous function $\tilde{f}$ on a G_δ set G with $S \subseteq G \subseteq \overline{S}$.

Proof. Let $G := \overline{S} \cap \{x \in X \,|\, \operatorname{osc}_f(x) = 0\}$. Clearly $S \subseteq G$ as f is continouos on f.

Claim 1. *G is G_δ*

Subproof. $\overline{S}$ is closed and

$$\bigcap_{n\geqslant 1}\{x : \operatorname{osc}_f(x) < \frac{1}{n}\}$$

is an intersection of open sets. ■

is an intersection of open sets.

For $x \in G$, as $x \in \overline{S}$, there exists $(x_n)_{x_n<\omega}$, $x_n \in S$ such that $x_n \to x$. We have that $(f(x_n))_n$ is Cauchy, as $\operatorname{osc}_f(x) = 0$.

Something is missing here

□

Corollary A.5. Let X be Polish and $Y \subseteq X$ Polish. Then Y is G_δ.

Proof. □

TODO

A.2.4 Exercise 4

Define

$$\begin{aligned} f\colon \omega^\omega &\longrightarrow 2^\omega \\ (x_n) &\longmapsto 0^{x_0}10^{x_1}1\dots. \end{aligned}$$

(1) f is a topological embedding: Consider a basic open set $B = \prod_{i<n} X_i \times \omega^\omega$ for some $X_i \subseteq \omega$.

Then $f(B) = \left(\bigcup_{x\in\prod_{i<n} X_i} B_x\right) \cap f(\omega^\omega)$ is open in $f(\omega^\omega)$, where $B_x := \{0^{x_0}10^{x_1}1\dots 10^{x_n-1}1\} \times 2^\omega$.

On the other hand let $C = \{x_0x_1x_2x_3x_4\dots x_{n-1}\} \times 2^\omega$ be some basic open set of 2^ω. W.l.o.g. $x_0x_1\dots x_{n-1}$ has the form $0^{a_0}10^{a_1}1\dots 10^{a_k}x_{n-1}$. If $x_{n-1} = 1$, we get

$$f^{-1}(C \cap f(\omega^\omega)) = \{(a_0, a_1, \dots, a_k)\} \times \omega^\omega.$$

In the case of $x_{n-1} = 0$, it is

$$f^{-1}(C \cap f(\omega^\omega)) = \bigcup_{b>a_k} \{(a_0, a_1, \dots, a_{k-1}, b)\} \times \omega^\omega.$$

In both cases the preimage is open.

(2) $C := 2^\omega \backslash f(\omega^\omega)$ is countable and dense in 2^ω.

We have $C = \{x \in 2^\omega | x_i = 0 \text{ for all but finitely many } i\} = \bigcup_{i<\omega}(2^i \times 1^\omega)$. Clearly this is countable.

For denseness take some $x \in 2^\omega$. Let $x^{(n)}$ be defined by $x_i^{(n)} = x_i$ for $i < n$ and $x_i^{(n)} = 0$ for $i \geqslant n$. Then $x^{(n)} \in C$ for all n, and $x^{(n)}$ converges to x.

(3) $f(\omega^\omega)$ is G_δ:

We have

$$\begin{aligned} f(\omega^\omega) &= 2^\omega \backslash \left(\bigcup_{i<\omega}(2^i \times 1^\omega)\right) \\ &= \bigcap_{i<\omega}\left(2^\omega \backslash (2^i \times 1^\omega)\right). \end{aligned}$$

(4) C as in (2) is homeomorphic to $\mathbb{Q}$.

Go to the right in the even digits, go to the left for the odd digits, i.e. let $C = (1, -1, 1, -1, \dots)$ and set $x < y$ iff $C \cdot x <_{\text{lex}} C \cdot y$, where $<_{\text{lex}}$ denotes the lexicographical ordering. Note that the order topology of $<$ on C agrees with the subspace topology from 2^ω.

By Cantor's theorem for countable, unbounded, dense linear linear orders, we get an order isomorphism $C \leftrightarrow \mathbb{Q}$. This is also a homeomorphism, as the topologies on C and $\mathbb{Q}$ are the respective order topologies.

A.3 Sheet 3

[Tutorial 04,]

A.3.1 Exercise 1

Let $A \neq \varnothing$ be discrete. For $D \subseteq A^\omega$, let

$$T_D := \{x|_n \in A^{<\omega} | x \in D, n \in \mathbb{N}\}.$$

(a) For any $D \subseteq A^\omega$, T_D is a pruned tree:

Clearly T_D is a tree. Let $x \in T_D$. Then there exists $d \in D$ such that $x = d|_n$. Hence $x \subseteq d|_{n+1} \in T_D$. Thus x is not a leaf, i.e. T_D is pruned.

(b) For any $T \subseteq A^{<\omega}$, $[T]$ is a closed subset of A^ω:

Let $a \in A^\omega \backslash [T]$. Then there exists some n such that $a|_n \notin T$. Hence $\{a_0\} \times \ldots \times \{a_{n+1}\} \times A^\omega$ is an open neighbourhood of a disjoint from $[T]$.

(c) $T \mapsto [T]$ is a bijection between the pruned trees on A and the closed subsets of A^ω.

Claim 3. *$[T_D] = D$ for any closed subset $D \subseteq A^\omega$.*

Subproof. Clearly $D \subseteq [T_D]$. Let $x \in [T_D]$. Then for every $n < \omega$, there exists some $d_n \in D$ such that $d_n|_n = x|_n$. Clearly the d_n converge to x. Since D is closed, we get $x \in D$. ■

This shows that $T \mapsto [T]$ is surjective.

Now let $T \neq T'$ be pruned trees. Then there exists $x \in T \triangle T'$, wlog. $x \in T \backslash T'$. Since T is pruned by applying the axiom of countable choice we get an infinite branch $x' \in [T] \backslash [T']$. Hence the map is injective.

(d) Let $N_s := \{x \in A^\omega | s \subseteq x\}$. Show that every open $U \subseteq A^\omega$ can be written as $U = \bigcup_{s \in S} N_s$ for some set of pairwise incompatible $S \subseteq A^{<\omega}$.

Let U be open. Then U has the form

$$U = \bigcup_{i \in I} X_i \times A^\omega$$

for some $X_i \subseteq A^{n_i}$, $n_i < \omega$. Clearly $U = \bigcup_{s \in S'} N_s$ for $S' := \bigcup_{i \in I} X_i$. Define

$$S := \{s \in S' | \neg \exists t \in S'.\ t \subseteq s \wedge |t| < |s|\}.$$

Then the elements of S are pairwise incompatible and $U = \bigcup_{s \in S} N_s$.

(e) Let $T \subseteq A^{<\omega}$ be an infinite tree which is finitely splitting. Then $[T]$ is nonempty:

Let us recursively construct a sequence of compatible $s_n \in T$ with $|s_n| = n$ such that $\{s_n\} \times A^{<\omega} \cap T$ is infinite. Let s_0 be the empty sequence; by assumption T is infinite. Suppose that s_n has been chosen. Since T is finitely splitting, there are only finitely many $a \in A$ with $s_n \frown a \in T$. Since $\{s_n\} \times A^{<\omega} \cap T$ is infinite, there must exist at least on $a \in A$ such that $\{s_n \frown a\} \times A^{<\omega} \cap T$ is infinite. Define $s_{n+1} := s_n \frown a$.

Then the union of the s_n is an infinite branch of T, i.e. $[T]$ is nonempty.

(f) Then $[T]$ is compact: TODO

A.3.2 Exercise 2

handwritten

A.3.3 Exercise 3

handwritten

A.3.4 Exercise 4

Notation A.5.62. For $A \subseteq X$ let A' denote the set of accumulation points of A.

Theorem A.6. Let X be a Polish space. Then there exists a unique partition $X = P \sqcup U$ of X into a perfect closed subset P and a countable open subset U.

Proof. Let P be the set of condensation points of X and $U := X \setminus P$.

Claim 1. *U is open and countable.*

Subproof. Let S be a countable dense subset. For each $x \in U$, there is an $\varepsilon_x > 0$, $s_x \in S$ such that $x \in B_{\varepsilon_x}(s_x)$ is at most countable. Clearly $B_{\varepsilon_x}(s_x) \subseteq U$, as for every $y \in B_{\varepsilon_x}(s_x)$, $B_{\varepsilon_x}(s_x)$ witnesses that $y \notin P$. Thus $U = \bigcup_{x \in U} B_{\varepsilon_x}(s_x)$ is open. Wlog. $\varepsilon_x \in \mathbb{Q}$ for all x. Then the RHS is the union of at most countably many countable sets, as $S \times \mathbb{Q}$ is countable. ■

Claim 2. *P is perfect.*

Subproof. Let $x \in P'$ and $x \in U$ an open neighbourhood. Then there exists $y \in P \cap U$. In particular, U is an open neighbourhood of y, hence U is uncountable. It follows that $x \in P$.

On the other hand let $x \in P$ and let U be an open neighbourhood. We need to show that $U \cap P \setminus \{x\}$ is not empty. Suppose that for all $y \in U \cap P \setminus \{x\}$, there is an open neighbourhood U_y such that U_y is at most countable. Wlog. $U_y = B_{\varepsilon_y}(s_y)$ for some $s_y \in S$, $\varepsilon_y > 0$, where S is again a countable dense subset. Wlog. $\varepsilon_y \in \mathbb{Q}$. But then

$$U = \{x\} \cup \bigcup_{y \in U} B_{\varepsilon_y}(s_y)$$

is at most countable as a countable union of countable sets, contradiction $x \in P$. ■

Claim 3. *Let P, U be defined as above and let $P_2 \subseteq X$, $U_2 \subseteq X$ be such that P_2 is perfect and closed, U_2 is countable and open and $X = P_2 \sqcup U_2$. Then $P_2 = P$ and $U_2 = U$.*

TODO

□

Corollary A.7. Any Polish space is either countable or has cardinality equal to $\mathfrak{c}$.

Subproof. Let $X = P \sqcup U$ where P is perfect and U is countable. If $P \neq \emptyset$, we have $|P| = \mathfrak{c}$ by Corollary 1.21. ■

A.4 Sheet 4

[Tutorial 05, 2023-11-14]

A.4.1 Exercise 1

(a) $\langle X_\alpha : \alpha \rangle$ is a descending chain of closed sets (transfinite induction).

Since X is second countable, there cannot exist uncountable strictly decreasing chains of closed sets:

Suppose $\langle X_\alpha, \alpha < \omega_1 \rangle$ was such a sequence, then $X \backslash X_\alpha$ is open for every α, Let $\{U_n : n < \omega\}$ be a countable basis. Then $N(\alpha) = \{n | U_n \cap (X \backslash X_\alpha) \neq \emptyset\}$, is a strictly ascending chain in ω.

(b) We need to show that X_{α_0} is perfect and closed. It is closed since all X_α are, and perfect, as a closed set F is perfect iff it coincides F'.

$X \backslash X_{\alpha_0}$ is countable: $X_\alpha \backslash X_{\alpha+1}$ is countable as for every x there exists a basic open set U, such that $U \cap X_\alpha = \{x\}$, and the space is second countable. Hence $X \backslash X_{\alpha_0}$ is countable as a countable union of countable sets.

A.4.2 Exercise 2

handwritten

A.4.3 Exercise 3

- Let $Y \subseteq \mathbb{R}$ be G_δ such that Y and $\mathbb{R} \backslash Y$ are dense in $\mathbb{R}$. Then $Y \cong \mathcal{N}$.

  Y is Polish, since it is G_δ.

  Y is 0-dimensional, since the sets $(a, b) \cap Y$ for $a, b \in \mathbb{R} \backslash Y$ form a clopen basis.

Each compact subset of Y has empty interior: Let $K \subseteq Y$ be compact and $U \subseteq K$ be open in Y. Then we can find cover of U that has no finite subcover $\lightning$.

- Let $Y \subseteq \mathbb{R}$ be G_δ and dense such that $\mathbb{R}\backslash Y$ is dense as well. Define $Z := \{x \in \mathbb{R}^2 | |x| \in Y\} \subseteq \mathbb{R}^2$. Then Z is dense in $\mathbb{R}^2$ and $\mathbb{R}^2\backslash\mathbb{Z}$ is dense in $\mathbb{R}^2$.

  We have that for every $y \in Y$ $\partial B_y(0) \subseteq Z$.

  Other example: Consider $\mathbb{R}^2\backslash\mathbb{Q}^2$.

A.4.4 Exercise 4

(a) Let d be a compatible, complete metric on X, wlog. $d \leqslant 1$. Set $U_\varnothing := X$. Suppose that U_s has already been chosen. Then $D_s := X\backslash U_s$ is closed. Hence $U_s^{(n)} := \{x \in X | \operatorname{dist}(x, D_s) > \frac{1}{n}\}$ is open. Let m be such that $D_s^{(m)} \neq \varnothing$. Clearly $\overline{U_s^{(n)}} \subseteq U_s$. Let $(B_k)_{k<\omega}$ be a countable cover of X consisting of balls of diameter $2^{-|s|-2}$. Take some bijection $\varphi\colon \omega \to \omega \times (\omega\backslash m)$ and set $U_{s^\frown i} := U_s^{(\pi_1(\varphi(i)))} \cap B_{\pi_2(\varphi(i))}$, where there π_i denote the projections (if this is empty, set $U_{s^\frown i} := U_s^{\pi_1(\varphi(j))} \cap B_{\pi_2(\varphi(j))}$ for some arbitarily chosen $j < \omega$ such that it is not empty). Then $\overline{U_{s^\frown i}} \subseteq \overline{U_s^{(n)}} \subseteq U_s$,

$$\bigcup_{i<\omega} U_{s^\frown i} = \bigcup_{n<\omega} U_s^{(n)} = U_s$$

and $\operatorname{diam}(U_{s^\frown i}) \leqslant \operatorname{diam}(B_{\pi_2(\varphi(i))})$.

(b) Let $s \in \omega^\omega$. Then

$$\bigcap_{n<\omega} \overline{U_{s|_n}}$$

contains exactly one point. Let f be the function that maps an $s \in \omega^\omega$ to the unique point in the intersection of the $\overline{U_{s|_n}}$. Let $x \in X$ be some point. Then by induction we can construct a sequence $s \in \omega^\omega$ such that $x \in U_{s|_n}$ for all n, hence $x = f(s)$, i.e. f is surjective.

Let $B \overset{\text{open}}{\subseteq} X$. Then $B = \bigcup_{i\in I} U_i$ for some $i \subseteq \omega^{<\omega}$, as every basic open set can be recovered as a union of U_i and $f^{-1}(B) = \bigcup_{i\in I} \left(\{i_0\} \times \ldots \{i_{|i|-1}\}\right) \times \omega^\omega$ is open, hence f is continuous.

On the other hand, consider an open ball $B := \{\prod_{i<n}\{x_i\}\} \times \omega^\omega \subseteq \omega^\omega$. Then $f(B) = U_{(x_0,\ldots,x_{n-1})}$ is open, hence f is open.

A.5 Sheet 5

[Tutorial 06,]

A.5.1 Exercise 1

Fact A.7.63. X is Baire iff every non-empty open set is non-meager.

In particular, let X be Baire, then $U \overset{\text{open}}{\subseteq} X$ is Baire.

(a) Let X be a non-empty Baire space and let $A \subseteq X$. Show that A cannot be both meager and comeager.

Suppose that $A \subseteq X$ is meager and comeager. Then $A = \bigcup_{n<\omega} U_n$ and $X \setminus A = \bigcup_{n<\omega} V_n$ for some nwd sets U_n, V_n. Then $X = A \cup (X \setminus A)$ is meager. Let $X = \bigcup_{n<\omega} W_n$ be a union of nwd sets. Wlog. the W_n are closed (otherwise replace them $\overline{W_n}$) Then $\varnothing = \bigcap_{n<\omega}(X \setminus W_n)$ is a countable intersection of open, dense sets, hence dense $\lightning$

(b) Let X be a topological space. The relation $=^*$ is transitive:

Suppose $A =^* B$ and $B =^* C$. Then $A \triangle C \subseteq (A \triangle B \cup B \triangle C)$ is contained in a meager set. Since a subset of a nwd set is nwd, a subset of a meager set is meager. Hence $A \triangle C$ is meager, thus $A =^* C$.

(c) Let X be a topological space. Let $A \subseteq X$ be a set with the Baire property, then at least one of the following hold:

(i) A is meager,

(ii) there exists $\varnothing = U \overset{\text{open}}{\subseteq} X$ such that $A \cap U$ is comeager in U.

Suppose there was $A \subseteq X$ such that (i) does not hold. Then there exists $U \overset{\text{open}}{\subseteq} X$ such that $A =^* U$. In particular, $A \triangle U$ is meager, hence $U \cap (A \triangle U) = U \setminus A$ is meager. Thus $A \cap U$ is comeager in U.

Now suppose that X is a Baire space. Suppose that for A (i) and (ii) hold. Let $\varnothing \neq U \overset{\text{open}}{\subseteq} X$ be such that $A \cap U$ is comeager in U. Since U is a Baire space, this contradicts (a).

A.5.2 Exercise 2

Let $(U_i)_{i<\omega}$ be a countable base of Y. We want to find a G_δ set $A \subseteq X$ such that $f|_A$ is continuous. It suffices make sure that $f^{-1}|_A(U_i)$ is open for all $i < \omega$. Take some $i < \omega$. Then $V_i \setminus M_i \subseteq f^{-1}(U_i) \subseteq V_i \cup M_i$, where V_i is open and M_i is meager. Let $M_i' \supseteq M_i$ be a meager F_σ-set. Now let $A := X \setminus \bigcup_{i<\omega} M_i'$. We have that A is a countable intersection of open dense sets, hence it is dense and G_δ. For any $i < \omega$, $V_i \cap A \subseteq f|_A^{-1}(U_i) \subseteq (V_i \cup M_i) \cap A = V_i \cap A$, so $f|_A^{-1}(U_i) = V_i \cap A$ is open.

A.5.3 Exercise 3

handwritten

A.5.4 Exercise 4

Lemma A.8. There exists a non-meager subset $A \subseteq \mathbb{R}^2$ such that no three points of A are collinear.

This requires the use of the axiom of choice.

Proof. Enumerate the continuum-many F_σ subsets of $\mathbb{R}^2$ as $(F_i)_{i<\mathfrak{c}}$. We will inductively construct a sequence $(a_i)_{i<\mathfrak{c}}$ of points of $\mathbb{R}^2$ such that for each $i < \mathfrak{c}$:

(i) $\{a_j | j \leqslant i\}$ is not a subset of F_i and

(ii) $\{a_j | j \leqslant i\}$ does not contain any three collinear points.

(a) Let $B = \{x \in \mathbb{R} | (F_i)_x \text{ is meager}\}$. Then B is comeager in $\mathbb{R}$ and $|B| = \mathfrak{c}$.

    We have $|B| = \mathfrak{c}$: B contains a comeager G_δ set, say B'. B' is Polish, hence $B' = P \cup C$ for P perfect and C countable, and $|P| \in \{\mathfrak{c}, 0\}$. But B' can't contain an isolated point.

(b) We use B to find a suitable point a_i:

    To ensure that (i) holds, it suffices to chose $a_i \notin F_i$. Since $|B| = \mathfrak{c}$ and $|\{a_i | j < i\}| = |i| < \mathfrak{c}$, there exists some $x \in B \setminus \{\pi_1(a_j) | j < i\}$, where π denotes the projection. Choose one such x. We need to find $y \in \mathbb{R}$, such that $(x, y) \notin F_i$ and $\{a_j | j < i\} \cup \{(x, y)\}$ does not contain three collinear points.

    Since $(F_i)_x$ is meager, we have that $|\{x\} \times \mathbb{R} \setminus F_i| = |\mathbb{R} \setminus (F_i)_x| = \mathfrak{c}$. Let $L := \{y \in \mathbb{R} | \exists j < k < i.\ a_j, a_k, (x, y) \text{are collinear}\}$. Since every pair $a_j \neq a_k, j < k < i$, adds at most one point to L, we get $|L| \leqslant |i|^2 < \mathfrak{c}$. Hence $|\mathbb{R} \setminus (F_i)_x \setminus L| = \mathfrak{c}$. In particular, the set is non empty, and we find y as desired and can set $a_i := (x, y)$.

(c) A is by construction not a subset of any F_σ meager set. Hence it is not meager, since any meager set is contained in an F_σ meager set.

□

(d) For every $x \in \mathbb{R}$ we have that A_x contains at most two points, hence it is meager. In particular $\{x \in \mathbb{R} | A_x \text{ is meager}\} = \mathbb{R}$ is comeager. However A is not meager. Hence A can not be a set with the Baire property by Kuratowski-Ulam (2.9). In particular, the assumption of the set having the BP is necessary.

A.6 Sheet 6

[Tutorial 07, 2023-11-28]

A.6.1 Exercise 1

Warning A.9. Note that not every set has a density!

(a) Let $X = \mathbb{I}^\omega$. Let $C_0 = \{(x_n) : x_n \to 0\}$. Claim: $C_0 \in \Pi^0_3(X)$ (intersections of F_σ sets).

We have
$$x \in C_0 \iff \forall q \in \mathbb{Q}^+.\ \exists N.\ \forall n \geqslant N.\ x_n \leqslant q,$$
i.e.
$$C_0 = \bigcap_{q \in \mathbb{Q}^+} \bigcup_{N<\omega} \bigcap_{n>N} \{x_n : x_n \leqslant q\}.$$

Clearly this is a Π^0_3 set.

(b) Let $Z := \{f \in 2^\omega : f(\mathbb{N}) \text{ has density } 0\}$. Claim: $Z \in \Pi^0_3(2^\mathbb{N})$. It is
$$Z = \bigcap_{q \in \mathbb{Q}^+} \bigcup_{N<\omega} \bigcap_{n \geqslant N} \{f \in 2^\omega : \frac{\sum_{i<n} f(i)}{n} \leqslant nq\}.$$

Clearly this is a Π^0_3-set.

A.6.2 Exercise 2

Recall Theorem 3.7:

> **Fact A.9.64.** Let (X, τ) be a Polish space and $A \in \mathcal{B}(X)$. Then there exists $\tau' \supseteq \tau$ with the same Borel sets as τ such that A is clopen.

(Do it for A closed, then show that the sets which work form a σ-algebra).

(a) Let (X, τ) be Polish. We want to expand τ to a Polish topology τ_0 maintaining the Borel sets, such that (X, τ') is 0d.

Let $(U_n)_{n<\omega}$ be a countable base of (X, τ). Each U_n is open, hence Borel, so by a theorem from the lecture™ there exists a Polish topology τ_n such that U_n is clopen, preserving Borel sets.

Hence we get τ_∞ such that all the V_n are clopen in τ_∞. Let $\tau^1 := \tau_\infty$. Do this ω-many times to get τ^ω. τ^ω has a base consisting of finite intersections $A_1 \cap \ldots \cap A_n$, where A_i is a basis element we chose to construct τ_i, hence clopen.

(b) Let $(X, \tau_X), Y$ be Polish and $f : X \to Y$ Borel. Show $\exists \tau' \supseteq \tau$ maintaining the Borel structure with f continuous.

Let $(U_n)_n$ be a countable base of Y. Clopenize all the preimages of the $(U_n)_n$.

(c) Let $f : X \to Y$ be a Borel isomorphism. Then there are finer topologies preserving the Borel structure such that $f : X' \to Y'$ is a homeomorphism.

Repeatedly apply (c). Get τ^1_X to make f continuous. Then get τ^1_Y to make f^{-1} continuous (possibly violating continuity of f) and so on.

Let $\tau^\omega_X := \langle \tau^n_X \rangle$ and similarly for τ^ω_Y.

Idea. *If you do something and it didn't work, try doing it again (ω-many times).*

A.6.3 Exercise 3

(a) Show that if Γ is self-dual (closed under complements) and closed under continuous preimages, then for any topological space X, there does not exist an X-universal set for $\Gamma(X)$.

Suppose there is an X-universal set for $\Gamma(X)$, i.e. $U \subseteq X \times X$ such that $U \in \Gamma(X \times X) \wedge \{U_x :\in X\} = \Gamma(X)$.

Consider $X \xrightarrow[x \mapsto (x,x)]{d} X \times X$.

Let $V = U^c$. Then $V \in \Gamma(X \times X)$ and $d^{-1}(V) \in \Gamma(X)$. Then $d^{-1}(V) = U_x$ for some x. But then $(x, x) \in U \iff x \in d^{-1}(V) \iff (x, x) \notin U$ ↯.

(b) Let ξ be an ordinal and let X be a topological space. Show that neither $\mathcal{B}(X)$ nor $\Delta^0_\xi(X)$ can have X-universal sets.

Clearly $\mathcal{B}(X)$ is self-dual and closed under continuous preimages. Clearly $\Delta^0_\xi(X)$ is self-dual and closed under continuous preimages (by a trivial induction).

A.6.4 Exercise 4

Recall:

Fact A.9.65 (Sheet 5, Exercise 1)**.** Let $\varnothing \neq X$ be a Baire space. Then $\forall A \subseteq X$, A is either meager or locally comeager.

Theorem A.10. [a] Let X, Y be Polish.

For $\varnothing \neq U \overset{\text{open}}{\subseteq} Y$ let

$$A_U := \{x \in X : A_x \text{ is not meager in } U\}.$$

Define

$$\mathcal{A} := \{A \in \mathcal{B}(X \times Y) : \forall \varnothing \neq U \overset{\text{open}}{\subseteq} Y.\ A_U \text{ is Borel}\}.$$

Then $\mathcal{A}$ contains all Borel sets.

[a] See Kechris 16.1

Proof. (i) Show for $V \in \mathcal{B}(X), W \overset{\text{open}}{\subseteq} Y$ that $V \times W \in \mathcal{A}$.

Clearly $V \times W$ is Borel and $\{x \in X : W \cap U \text{ is not meager}\} \in \{\varnothing, V\}$.

(ii) Let $(A_n)_{n<\omega} \in \mathcal{A}^\omega$. Then $\bigcap_n A_n \in \mathcal{A}$. $((\bigcup_n A_n)_U = \bigcup_n (A_n)_U)$.

(iii) Let $A \in \mathcal{A}$ and $B = A^c$. Fix $\emptyset \neq U \subseteq Y$. Then $\{x : A_x \text{is not meager in } U\}$ is Borel, i.e. $\{x : A_x^c \text{ is not meager in } U\}$ is Borel.

Since A is Borel, A_x is Borel as well. Hence by the fact:

$$\begin{aligned}
&\{x : A_x^c \text{ is not meager in } U\} \\
=&\{x\colon A_x^c \text{ is locally comeager in } U\} \\
=&\{x\colon \exists \emptyset \neq V \overset{\text{open}}{\subseteq} V.\ A_x \text{ is meager in } V\} \\
=&\bigcup_{\emptyset \neq V \overset{\text{open}}{\subseteq} U} A_V^c
\end{aligned}$$

(a countable union suffices, since we only need to check this for V of the basis; if $A \subseteq V$ is nwd, then $A \cap U \subseteq U$ is nwd for all $U \overset{\text{open}}{\subseteq} V$).

□

A.7 Sheet 7

[Tutorial 08, 2023-12-05]

A.7.1 Exercise 1

- For $\xi = 1$ this holds by the definition of the subspace topology.

  We now use transfinite induction, to show that the statement holds for all ξ. Suppose that $\Sigma^0_\zeta(Y)$ and $\Pi^0_\zeta(Y)$ are as claimed for all $\zeta < \xi$.

  Then

$$\begin{aligned}
\Sigma^0_\xi(Y) &= \{\bigcup_{n<\omega} A_n : A_n \in \Pi^0_{\alpha_n}(Y), \alpha_n < \xi\} \\
&= \{\bigcup_{n<\omega} (A_n \cap Y) : A_n \in \Pi^0_{\alpha_n}(X), \alpha_n < \xi\} \\
&= \{Y \cap \bigcup_{n<\omega} A_n : A_n \in \Pi^0_{\alpha_n}(X), \alpha_n < \xi\} \\
&= \{Y \cap A : A \in \Sigma^0_\xi(X)\}.
\end{aligned}$$

  and

$$\begin{aligned}
\Pi^0_\xi(Y) &= \neg\Sigma^0_\xi(Y) \\
&= \{Y \setminus A : A \in \Sigma^0_\xi(Y)\} \\
&= \{Y \setminus (A \cap Y) : A \in \Sigma^0_\xi(X)\} \\
&= \{Y \cap (X \setminus A) : A \in \Sigma^0_\xi(X)\} \\
&= \{Y \cap A : A \in \Pi^0_\xi(X)\}.
\end{aligned}$$

- Let $V \in \mathcal{B}(Y)$.

We show that $f^{-1}(V) \in \mathcal{B}(Y)$, by induction on the minimal ξ such that $V \in \Sigma^0_\xi$. For $\xi = 0$ this is clear. Suppose that we have already shown $f^{-1}(V') \in \mathcal{B}(Y)$ for all $V' \in \Sigma^0_\zeta$, $\zeta < \xi$. Then $f^{-1}(Y \setminus V') = X \setminus f^{-1}(V') \in \mathcal{B}(V)$, since complements of Borel sets are Borel. In particular, this also holds for Π^0_ζ sets and $\zeta < \xi$. Let $V \in \Sigma^0_\xi$. Then $V = \bigcap_n V_n$ for some $V_n \in \Pi^0_{\alpha_n}$, $\alpha_n < \xi$. In particular $f^{-1}(V) = \bigcup_n f^{-1}(V_n) \in \mathcal{B}(X)$.

A.7.2 Exercise 2

Recall **??**.

Let $(A_i)_{i<\omega}$ be analytic subsets of a Polish space X. Then there exists Polish spaces Y_i and $f_i \colon Y_i \to X$ continuous such that $f_i(B_i) = A_i$ for some $B_i \in \mathcal{B}(Y_i)$.

- $\bigcup_i A_i$ is analytic: Consider the Polish space $Y := \coprod_{i<\omega} Y_i$ and $f := \coprod_i f_i$, i.e. $Y_i \ni y \mapsto f_i(y)$. f is continuous, $\coprod_{i<\omega} B_i \in \mathcal{B}(Y)$ and
$$f(\coprod_{i<\omega} B_i) = \bigcup_i A_i.$$
- $\bigcap_i A_i$ is analytic: Let $Z = \prod Y_i$ and let $D \subseteq Z$ be defined by
$$D := \{(y_n) : f_i(y_i) = f_j(y_j) \ \forall i,j\}.$$
D is closed, at it is the preimage of the diagonal under $Z \xrightarrow{(f_0,f_1,\ldots)} X^{\mathbb{N}}$. Then $\bigcap A_i$ is the image of D under $Z \xrightarrow{(y_n)\mapsto f_0(y_0)} X$.

  Other solution:

  Let $F_n \subseteq X \times \mathcal{N}$ be closed, and $C \subseteq X \times \mathcal{N}^{\mathbb{N}}$ defined by
$$C := \{(x, (y^{(n)})) : \forall n.\ (x, y^{(n)}) \in F_n\}.$$
C is closed and $\bigcap A_i = \operatorname{proj}_X(C)$.

A.7.3 Exercise 3

Lemma A.11. Let X be a second-countable topological space. Then every base of X contains a countable subset which is also a base of X.

Proof. Let $\mathcal{C} = \{C_n : n < \omega\}$ be a countable base of X and let $\mathcal{B} = \{B_i : i \in I\}$ be a base of X with (possibly uncountable) index set I.

Fix $n < \omega$. It suffices to show that C_n is a union of countably many elements of $\mathcal{B}$. As $\mathcal{B}$ is a base, $C_n = \bigcup_{j\in J} B_j$ for some $J \subseteq I$. Since $\mathcal{C}$ is a base, there exists $M_j \subseteq \mathbb{N}$ such that $B_j = \bigcup_{m \in M_j} C_m$ for all $j \in J$. Let $M = \bigcup_{j\in J} M_j \subseteq \mathbb{N}$. For each $m \in M$, there exists $f(m) \in J$ such that $m \in M_{f(m)}$. Then $\bigcup_{m\in M} B_{f(m)} = C_n$. $\square$

Remark A.11.66. We don't actually need this.

- We use the same construction as in exercise 2 (a) on sheet 6. Let $A \subseteq X$ be analytic, i.e. there exists a Polish space Y and $f\colon Y \to X$ Borel with $f(Y) = X$. Then f is still Borel with respect to the new topology, since Borel sets are preserved and by exercise 1 (b).

- Suppose that there exist no disjoint clopen sets U_0, U_1, such that $W \cap U_0$ and $W \cap U_1$ are uncountable.

  Let $W_0 := W$ Then there exist disjoint clopen sets $C_i^{(0)}$ such that $W_0 \subseteq \bigcup_{i<\omega} C_i^{(0)}$ and $\operatorname{diam}(C_i) < 1$, since X is zero-dimensional.

  By assumption, exactly one of the $C_i^{(0)}$ has uncountable intersection with W_0. Let i_0 be such that $W_0 \cap C_{i_0}^{(0)}$ is uncountable and set $W_1 := W_0 \cap C_{i_0}^{(0)}$. Note that $W_0 \backslash W_1 = \bigcup_{i \neq i_0} C_i^{(0)}$ is countable.

  Let us recursively continue this construction: Suppose that W_n uncountable has been chosen. Then choose $C_i^{(n)}$ clopen, disjoint with diameter $\leqslant \frac{1}{n}$ such that $W_n \subseteq \bigcup_i C_i^{(n)}$ and let i_n be the unique index such that $W_n \cap C_{i_n}^{(n)}$ is uncountable.

  Since $\operatorname{diam}(C_{i_n}^{(n)}) \xrightarrow{n\to\infty} 0$ and the $C_{i_n}^{(n)}$ are closed, we get that $\bigcap_n C_{i_n}^{(n)}$ contains exactly one point. Let that point be x.

  However then

  $$W = \left(\bigcup_{n<\omega} \bigcup_{i \neq i_n} (C_i^{(n)} \cap W)\right) \cup \bigcap_n (W \cap C_{i_n}^{(n)}) = \left(\bigcup_{n<\omega} \bigcup_{i \neq i_n} (C_i^{(n)} \cap W)\right) \cup \{x\}$$

  is countable as a countable union of countable sets $\lightning$.

  Other proof (without using the existence of a countable clopen basis):

  We can cover X by countably many clopen sets of diameter $< \frac{1}{n}$: Cover X with open balls of diameter $< \frac{1}{n}$. Write each open ball as a union of clopen sets. That gives us a cover by clopen sets of diameter $< \frac{1}{n}$. As X is Lindelöf, there exists a countable subcover. Then continue as in the first proof.

- Note that this step does not help us to prove the statement. It was an error on the exercise sheet.

  Clearly this defines a Cantor scheme.

- Let Y be a Polish space and $A \subseteq Y$ analytic and uncountable. Expand the topology on Y so that Y is zero dimensional and A is still analytic.

  Then there exists a Polish space X and a continuous function $f\colon X \to Y$ such that $f(X) = A$.

A is uncountable, so by (2) there exists non-empty disjoint clopen V_0, V_1 such that $V_0 \cap A$ and $V_1 \cap A$ are uncountable.

Let $W_0 = f^{-1}(V_0 \cap A)$ and $W_1 = f^{-1}(V_1 \cap A)$. W_0 and W_1 are clopen and disjoint. We can cover W_0 with countably many open sets of diameter $\leqslant \frac{1}{n}$ and similarly for W_1. Then pick open sets such that there image is uncountable.

Repeating this construction we get a Cantor scheme on X. So we get $2^{\mathbb{N}} \overset{s}{\hookrightarrow} X$ and by construction of the cantor scheme, we get that $f \circ s$ is injective and continuous.

A.7.4 Exercise 4

Proof of Schröder-Bernstein:

Let $X_0 := X$, $Y_0 := Y$ and define $X_{i+1} := g(Y_i)$, $Y_{i+1} := f(X_i)$. We have $X_{i+1} \subseteq X_i$ and similarly for Y_i. f and g are bijections between $X_\omega := \bigcap X_i$ and $Y_\omega := \bigcap Y_i$.

$$
\begin{tikzcd}
X\setminus X_\omega = & (X_0\setminus X_1) \arrow[drr, "f"] & \cup & (X_1\setminus X_2) & \cup & (X_2\setminus X_3) \arrow[dr, "f"] & \cdots \\
Y\setminus Y_\omega = & (Y_0\setminus Y_1) \arrow[urr, "g"] & \cup & (Y_1\setminus Y_2) & \cup & (Y_2\setminus Y_3) \arrow[ur, "g"] & \cdots
\end{tikzcd}
$$

By Theorem 3.23 the injective image via a Borel function of a Borel set is Borel.

Theorem 3.23 also gives that the inverse of a bijective Borel map is Borel. So we can just do the same proof and every set will be Borel.

[Tutorial 09, 2023-12-12]

Fact A.11.67. Let X, Y be topological spaces X (quasi-)compact and Y Hausdorff. Let $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism.

Proof. Compact subsets of Hausdorff spaces are closed. □

A.8 Sheet 8

Material on topological dynamics:

- Terence Tao's notes on ergodic theory 254A: [Tao08]
- [Fur63] (uses very different notation!).

A.8.1 Exercise 1

Remark A.11.68. Σ_1^1-complete sets are in some sense the "worst" Σ_1^1-sets: Deciding whether an element is contained in the Σ_1^1-complete set is at least as "hard" as as for any Σ_1^1 set.

> In particular, Σ^1_1-complete sets are not Borel.

Similarly as in Proposition 3.30 it can be shown that $L \in \Sigma^1_1(X)$: Consider $\{(x, \beta) \in X \times \mathcal{N} : \forall n.\ x_{\beta_n} | x_{\beta_{n+1}}\}$. This is closed in $X \times \mathcal{N}$, since it is a countable intersection of clopen sets and $L = \mathrm{proj}_X(D)$.

Since $\mathrm{IF} \subseteq \mathrm{Tr}$ is Σ^1_1-complete, it suffices to find a Borel map $f\colon \mathrm{Tr} \to X$ such that $x \in \mathrm{IF} \iff f(x) \in L$. Let $\varphi\colon \omega^2 + \omega \to \omega$ be bijective and let p_i denote the i-th prime. Define

$$\begin{aligned} \psi\colon \omega^{<\omega} &\longrightarrow \omega\backslash\{0\} \\ (s_0, s_1, \ldots, s_{n-1}) &\longmapsto \prod_{i<n} p_{\varphi(\omega \cdot i + s_i)}. \end{aligned}$$

Note that ψ is injective and that $s \in \omega^{<\omega}$ is an initial segment of $t \in \omega^{<\omega}$ iff $\psi(s)|\psi(t)$. Let

$$\begin{aligned} f'\colon \mathrm{Tr} &\longrightarrow \mathcal{P}(\omega\backslash\{0\}) \\ T &\longmapsto \{\varphi(s) : s \in T\}. \end{aligned}$$

We can turn this into a function $f\colon \mathrm{Tr} \to (\omega\backslash\{0\})^\omega$ by mapping a subset of $\omega\backslash\{0\}$ to the unique strictly increasing sequence whose elements are from that subset (appending $\varphi(\omega^2 + n), n \in \omega$, if the subset was finite). Note that $T \in \mathrm{IF} \iff f(T) \in L$. Furthermore f is Borel, since fixing a finite initial sequence (i.e. a basic open set of $(\omega\backslash\{0\})^\omega$) amounts to a finite number of conditions on the preimage.

A.8.2 Exercise 2

handwritten

A.8.3 Exercise 3

- $LO(\mathbb{N}) \overset{\text{closed}}{\subseteq} 2^{\mathbb{N}\times\mathbb{N}}$:

  We have $<\in LO(\mathbb{N})$ iff

  - $\forall x, y.\ (x \neq y \implies (x < y \vee x > y))$,
  - $\forall x.\ (x \not< x)$,
  - $\forall x, y, x.\ (x < y < z \implies x < z)$.

  Write this with $\bigcap$, i.e.

  $$\begin{aligned} LO(\mathbb{N}) = &\bigcap_{n\in\mathbb{N}} \{R : (n, n) \notin R\} \\ &\cap \bigcap_{m<n\in\mathbb{N}} (\{R : (n, m) \in R\} \cup \{R : (m, n) \in R\}) \\ &\cap \bigcap_{a,b,c\in\mathbb{N}} (\{R : (a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R\}. \end{aligned}$$

  This is closed as an intersection of clopen sets.

- We apply Theorem 3.16 (iv). Let $\mathcal{F} \subseteq LO(\mathbb{N}) \times \mathcal{N}$ be such that the $\mathcal{N}$-coordinate encodes a strictly decreasing sequence, i.e.

  $$(R, s) \in \mathcal{F} :\iff \forall n \in \mathbb{N}.\ (s(n+1), s(n)) \in R.$$

  We have that

  $$\mathcal{F} = \bigcap_{n \in \mathbb{N}} \{(R, s) \in LO(\mathbb{N}) \times \mathcal{N} : (s(n+1), s(n)) \in R\}$$

  is closed as an intersection of clopen sets.

  Clearly $\operatorname{proj}_{LO(\mathbb{N})}(\mathcal{F})$ is the complement of $WO(\mathbb{N})$, hence $WO(\mathbb{N})$ is co-analytic.

A.8.4 Exercise 4

Remark A.11.69. In the lecture we only look at metrizable flows, so the definitions from the exercise sheet and from the lecture don't agree.

Everywhere but here we will use the definition from the lecture.

- Consider

  $$\begin{aligned} \Phi \colon \mathbb{Z}\text{-flows on } X &\longrightarrow \operatorname{Homeo}(X) \\ (\alpha \colon \mathbb{Z} \times X \to X) &\longmapsto \alpha(1, \cdot) \\ \begin{pmatrix} \mathbb{Z} \times X & \longrightarrow & X \\ (z, x) & \longmapsto & \beta^z(x) \end{pmatrix} &\longleftarrow\!\shortmid \beta \in \operatorname{Homeo}(X). \end{aligned}$$

  Clearly this has the desired properties.

- Let X be a compact Polish space. What is the Borel complexity of $\operatorname{Homeo}(X)$ inside $\mathcal{C}(X, X)$?

  Recall that $\mathcal{C}(X, X)$ is a Polish space with the uniform topology. We have

  $$\begin{aligned} \operatorname{Homeo}(X) &= \{f \in \mathcal{C}(X, X) : f \text{ is bijective and } f^{-1} \text{ is continuous}\} \\ &= \{f \in \mathcal{C}(X, X) : f \text{ is bijective}\} \end{aligned}$$

  by the general fact

  Fact A.11.70. Let X be compact and Y Hausdorff, $f \colon X \to Y$ a continuous bijection. Then f is a homeomorphism.

[Tutorial 13, 2024-01-23]

Continuation of sheet 8, exercise 4.

Definition A.12. Let X be a compact metric space. For $K \subseteq X$ compact and $U \overset{\text{open}}{\subseteq} X$ let

$$S_{K,U} := \{f \in \mathcal{C}(X,X) : f(K) \subseteq U\}.$$

The **compact open topology** on $\mathcal{C}(X,X)$ is the topology that has $S_{K,U}$ as a subbase.

Fact A.12.71. If X is compact, then the compact open topology is the topology induced by the uniform metric d_∞.

Proof. Take some $S_{K,U}$. We need to show that this can be written as a union of open d_∞-balls. Let $f_0 \in S_{K,U}$. Consider the continuous function $d(-,U^c)$. Since $f_0(K)$ is compact, there exists $\varepsilon := \min d(f_0(K), U^c)$ and $B_\varepsilon(f_0) \subseteq S_{K,U}$.

On the other hand, consider $B_\varepsilon(f_0)$ for some $\varepsilon > 0$ and $f_0 \in \mathcal{C}(X,X)$.

As f_0 is uniformly continuous, there exists $\delta > 0$ such that $d(x,x') < \delta \implies d(f_0(x), f_0(x')) < \frac{\varepsilon}{3}$. Cover X with finitely many δ-balls $B_\delta(a_1), \ldots, B_\delta(a_k)$. Then

$$f_0(\overline{B_\delta(a_i)}) \subseteq \overline{f_0(B_\delta(a_i)} \subseteq \overline{B_{\frac{\varepsilon}{3}}(f_0(a_i))} \subseteq B_{\frac{\varepsilon}{2}}(f_0(a_i)).$$

For $i \leqslant k$, let $S_i := S_{\overline{B_\delta(a_i)}, B_{\frac{\varepsilon}{2}}(f_0(a_i))}$. Take $\bigcap_{i \leqslant k} S_i$. This is open in the compact open topology and $B_\varepsilon(f_0) \subseteq \bigcap_{i \leqslant k} S_i$. □

Claim 4. *$f \in \mathcal{C}(X,X)$ is surjective iff for all basic open $\varnothing \neq U \subseteq X$ there exists a basic open $\varnothing \neq V \subseteq X$ with $f(\overline{V}) \subseteq U$.*

Note that we can write this as a G_δ-condition.

Subproof. Take $B_\varepsilon(f(x_0)) \subseteq U$. Then there exists $\delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_{\frac{\varepsilon}{2}}(f(x_0))$ hence $f(\overline{B_\delta(x_0)}) \subseteq B_\varepsilon(f(x_0))$.

For the other direction take $y \in X$. We want to find a preimage. For every $B_{\frac{1}{n}}(y)$, there exists a basic open set V_n with $f(\overline{V}) \subseteq B_{\frac{1}{n}}(y)$. Take $x_n \in V_n$. Since X is compact, it is sequentially compact, so there exists a converging subsequence. Wlog. $x_n \to x$, so $f(x_n) \to f(x) = y$. ■

Claim 5. *$f \in \mathcal{C}(X,X)$ is injective iff for all basic open U,V with $\overline{U} \cap \overline{V} = \varnothing$ we have $f(\overline{U}) \cap f(\overline{V}) = \varnothing$.*

This is a G_δ-condition, since we can write it as

$$\bigcap_{U,V} S_{\overline{U}, f(\overline{V})^c}.$$

Subproof. $\implies$ is trivial.

$\impliedby$ follows since for all pairs $x, y \in X$, we can find $x \in U$, $y \in V$ such that $\overline{U} \cap \overline{V} = \emptyset$. ■

Hence $\mathrm{Homeo}(X, X)$ is G_δ. In particular it is a Polish space.

Let D be the set of $\mathbb{Z}$-flows with dense orbit.

Claim 6. *$f \in D \iff$ for all basic open $U, V \subseteq X$, there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.*

Subproof. Suppose that the orbit of $x_0 \in X$ is dense. Then there exist $k, l \in \mathbb{Z}$ such that $f^k(x_0) \in U$ and $f^l(x_0) \in V$, so $f^{l-k}U \cap V \neq \emptyset$.

For basic open sets V let

$$A_V := \{x \in X : \exists n.\ f^n(x) \in V\}.$$

By assumption, all the A_V are dense. Hence $\bigcap_V A_V$ is dense by the Baire Category Theorem (2.7).

$A_V = \bigcup_{n \in \mathbb{Z}} f^n(V)$ is open. ■

Claim 7. *The condition can be written as a G_δ set.*

Subproof. For $f_0(U) \cap V \neq \emptyset$ take $u \in U$ such that $f_0(u) \in V$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(f_0(u)) \subseteq U$, hence $B_\varepsilon(f_0)$ is an open neighbourhood contained in $\{f : f(U) \cap V \neq \emptyset\}$.

For $n = 2$ note that $d(f^2(u), f_0^2(u) \leqslant d(f(f(u)), f_0(f(u))) + d(f_0(f(u)), f_0(f_0(u)))$. The first part can be bounded by $d(f, f_0)$. For the second part, note that there exists δ such that

$$d(a, b) < \delta \implies d(f_0(a), f_0(b)) < \frac{\varepsilon}{2}.$$

Let $\eta := \min\{\delta, \frac{\varepsilon}{2}\}$ and consider $d_\infty(f, f_0) < \varepsilon$.

For other n it is some more work, which is left as an exercise. ■

[Tutorial 10, 2023-12-19]

A.9 Sheet 9

A.9.1 Exercise 1

$(X, \tau') \xrightarrow{x \mapsto x} (X, \tau)$ is Borel (by one of the equivalent definitions of being Borel). Thus $\mathcal{B}(X, \tau) \subseteq \mathcal{B}(X, \tau')$ (by the other equivalent definition of being Borel). Let $U \subseteq (X, \tau')$ be Borel. $\mathrm{id}|_U$ is injective, hence U is Borel in (X, τ) by Lusin-Suslin.

Related stuff

Fact A.12.72. Let X, Y be Polish. $f: X \to Y$ is Borel iff its graph Γ_f is Borel.

Proof. Take a countable open base $V_0, V_1, \ldots$ of Y. Then $\Gamma_f = \{(x, y) : \forall n < \omega.\ f(x) \in V_n \implies y \in V_n\}$ (because the space is Hausdorff). If f is Borel, then clearly the RHS is Borel since

$$\begin{aligned} &\{(x,y) : \forall n < \omega.\ f(x) \in V_n \implies y \in V_n\} \\ = &\bigcap_{n<\omega} (f^{-1}(V_n)^c Y \cup f^{-1}(V_n) \times V_n\} \end{aligned}$$

On the other hand suppose that Γ_f is Borel. Then

$$f^{-1}(B) = \pi_X(X \times B \cap \Gamma_f)$$

is analytic.[44] On the other hand

$$f^{-1}(B)^c = f^{-1}(B^c)$$

is analytic and we know that $\Sigma^1_1 \cap \Pi^1_1 = \mathcal{B}$ by the Corollary of the Lusin Separation Theorem (3.24). $\square$

In fact we have shown

Fact A.12.73. The following are equivalent

- f is Borel,
- Γ_f is Borel,
- Γ_f is analytic.

A.9.2 Exercise 2

Definition A.13. Let X be a topological space. Let $K(X)$ be the set of all compact subspaces of X. The **Vietoris Topology**, τ_V, on $K(X)$ is the topology with basic open sets

$$[U_0; U_1, \ldots, U_n] = \{K \in K(X) : K \subseteq U_0 \wedge \forall 1 \leqslant i \leqslant n.\ K \cap U_i \neq \varnothing\}$$

for $U_i \overset{\text{open}}{\subseteq} X$.

[44] Note that the projection of a Borel set is not necessarily Borel. Moreover note that we only used that Γ_f is analytic.

Definition A.14. Let (X, d) be a matric space with $d \leqslant 1$. We define a metric d_H on $K(X)$ as follows: $d_H(\varnothing, \varnothing) := 0$, $d_H(K, \varnothing) := 1$ for $K \neq \varnothing$ and

$$d_H(K_0, K_1) := \max\{\max_{x \in K_0} d(x, K_1), \max_{x \in K_1} d(x, K_0)\}$$

for $K_0, K_1 \neq \varnothing$.

Fact A.14.74. d_H is indeed a metric.

Proof. Let $\delta(K, L) := \max_{x \in K} d(x, L)$. It suffices to show $\delta(X, Z) \leqslant \delta(X, Y) + \delta(Y, Z)$, since then

$$\begin{aligned} d_H(X, Z) &\leqslant \max\{\delta(X, Y) + \delta(Y, Z), \delta(Z, Y) + \delta(Y, X)\} \\ &\leqslant d_H(X, Y) + d_H(Y, Z). \end{aligned}$$

Using the fact that $d(\cdot, Z)$ is uniformly continuous, specifically

$$|d(x, Z) - d(y, Z)| \leqslant d(x, y),$$

we get

$$\begin{aligned} d(x, Z) &\leqslant d(x, y) + d(y, Z) \\ &\leqslant d(x, y) + \delta(Y, Z) \\ \implies d(x, Z) - \delta(Y, Z) &\leqslant d(x, Y) \\ \implies \qquad d(x, Z) &\leqslant \delta(X, Y) + \delta(Y, Z) \\ \implies \qquad \delta(X, Z) &\leqslant \delta(X, Y) + \delta(Y, Z). \end{aligned}$$

□

- We have

$$\begin{aligned} d_H(K_0, K_1) < \varepsilon &\iff \max\{\max_{x \in K_0} d(x, K_1), \max_{x \in K_1} d(x, K_0)\} < \varepsilon \\ &\iff \max_{x \in K_0} d(x, K_1) < \varepsilon \wedge \max_{x \in K_1} d(x, K_0) < \varepsilon \\ &\iff K_0 \subseteq B_\varepsilon(K_1) \wedge K_1 \subseteq B_\varepsilon(K_0). \end{aligned}$$

- Note that a subbase of τ_V is given by $[U]$ and $\langle U \rangle := [X; U]$ for $U \overset{\text{open}}{\subseteq} X$.

  Let $K \in [U]$. Then $d(\cdot, U^c)\colon U \to \mathbb{R}_{\geqslant 0}$ is always non-zero and continuous. So $d(K, U^c)$ attains a minimum $\varepsilon > 0$. Then $B^H_\varepsilon(K) \subseteq U$, so $[U]$ is open in τ_V.

  Let $K \in \langle U \rangle$. Take some $k \in K \cap U$. Then there is some $\varepsilon > 0$ such that $B_\varepsilon(k) \subseteq U$. Then $K \in B^H_\varepsilon(K) \subseteq \langle U \rangle$.

Other direction

- Consider a countable dense subset of X. Let $\mathcal{K}$ be the set of finite subsets of that countable dense subset. Then $\mathcal{K} \subseteq K(X)$ is dense: Take $K \in K(X)$ an let $\varepsilon > 0$. K can be covered with finitely many ε-balls with centers from the countable dense subsets. Let $K' \in \mathcal{K}$ be the set of the centers. Then $d_H(K, K') \leqslant \varepsilon$.

A.9.3 Exercise 3

- By transfinite induction we get that α is an ordinal, since $\prec$ is well-founded and the supremum of a sets of ordinals is an ordinal. Since $\rho_\prec : X \to \alpha$ is a surjection, it follows that $\alpha \leqslant |X|$, i.e. $\alpha < |X|^+$.
- By induction on $\rho_{\prec_X}(x)$ we show that $\rho_{\prec_X}(x) \leqslant \rho_{\prec_Y}(f(x))$. For 0 this is trivial. Suppose that $\rho_{\prec_X}(x) = \alpha$ and the statement was shown for all $\beta < \alpha$. Then

$$\begin{aligned}\rho_{\prec_Y}(f(x)) &= \sup\{\rho_{\prec_Y}(y') + 1 | y' \prec f(x)\}\\ &\geqslant \sup\{\rho_{\prec_Y}(f(x')) + 1 | f(x') \prec f(x)\}\\ &\geqslant \sup\{\rho_{\prec_Y}(f(x')) + 1 | x' \prec x\}\\ &\geqslant \sup\{\rho_{\prec_X}(x') + 1 | x' \prec x\}\\ &= \rho_{\prec_X}(x).\end{aligned}$$

- Infinite branches of $T_\prec$ correspond to infinite descending chain of $\prec$, hence $T_\prec$ is well-founded iff $\prec$ is well-founded.

  Suppose that $\prec$ is well-founded. Note that $\rho_T(s)$ depends only on the last element of s, as for $s, s' \in T$ with the same last element, we have $s ^\frown x \in T \iff s' {}^\frown x \in T$.

  Let $s = (s_0, \ldots, s_n)$. Let us show that $\rho_T(s) = \rho_\prec(s_n)$. We use induction on $\rho_T(s)$. For leaves this is immediate. From the last exercise sheet we know that
$$\rho_T(s) = \sup\{\rho_T(s ^\frown a) + 1 | s ^\frown a \in T\}.$$

  Hence

$$\begin{aligned}\rho_T(s) &= \sup\{\rho_T(s ^\frown a) + 1 | s ^\frown a \in T\}\\ &= \sup\{\rho_\prec(a) + 1 | s ^\frown a \in T\}\\ &= \sup\{\rho_\prec(a) + 1 | a \prec s_n\}\\ &= \rho_\prec(s_n).\end{aligned}$$

A.9.4 Exercise 4

A solution can be found in [Ros12].

[Tutorial 11, 2024-01-09]

An equivalent definition of subflows can be given as follows:

Definition A.15. Let (X,T) be a flow with action α_x. Let $Y \subseteq X$ be a compact subspace of X. If Y is invariant under α_x, we say that (Y,T) (with action $\alpha_x|_{T\times Y}$ is a subflow of (X,T).

Example A.16 (Flows with a non-closed orbit)**.** 1. Consider $(S^1,\mathbb{Z})$ with action given by $1\cdot x = x + c$ for a fixed $c \in \mathbb{R}\backslash\mathbb{Q}$.[a] Then the orbit of 0, $\{nc : n \in \mathbb{Z}\}$ is dense but consists only of irrationals (except 0), so it is not closed.

2. Consider $(S^1,\mathbb{Q})$ with action $qx := x + q$. The orbit of 0, $\mathbb{Q}/\mathbb{Z} \subseteq S^1$, is dense but not closed.

   $(S^1,\mathbb{Q})$ is minimal.

[a] We identify S^1 and $\mathbb{R}/\mathbb{Z}$.

Example A.17 (**Left Bernoulli shift**)**.** Consider $(\{0,1\}^{\mathbb{Z}},T)$, where $T = \mathbb{Z}$ and the action is given by

$$\begin{aligned} \mathbb{Z}\times\{0,1\}^{\mathbb{Z}} &\longrightarrow \mathbb{Z} \\ (m,(x_n)_{n\in\mathbb{Z}}) &\longmapsto (x_{n+m})_{n\in\mathbb{Z}}. \end{aligned}$$

The orbit of $z := (0)_{n\in\mathbb{Z}}$ consist of only on point. In particular it is closed.

Let $x := ([n = 0])_{n\in\mathbb{Z}}$. Then $Tx = \{([n = m])_{n\in\mathbb{Z}} | m \in \mathbb{Z}\}$. Clearly $z \notin Tx$.

Claim 8. $z \in \overline{Tx}$

Proof. Consider a basic open $z \in U_I = \{y : y_i = 0, i \in I\}$ where $I \subseteq \mathbb{Z}$ is finite. Then $U_I \cap Tx \neq \varnothing$ as we can shift the 1 out of I, i.e. $(\max I + 1)x \in U_I$. □

Flows are always on non-empty spaces X.

Fact A.17.75. Consider a flow (X,T). The following are equivalent:

(i) Every T-orbit is dense.

(ii) There is no proper subflow,

If these conditions hold, the flow is called **minimal**.

Proof. (i) $\implies$ (ii): Let (Y,T) be a subflow of (X,T). take $y \in Y$. Then Ty is dense in X. But $Ty \subseteq Y$, so Y is dense in X. Since Y is closed, we get $Y = X$.

(ii) $\implies$ (i): Take $x \in X$. Consider Tx. It suffices to show that $\overline{Tx}$ is a subflow. Clearly $\overline{Tx}$ is closed, so it suffices to show that it is T-invariant. Let $y \in \overline{Tx}$

and $t \in T$. Take $ty \in U \overset{\text{open}}{\subseteq} X$. Since t^{-1} acts as a homeomorphism we have $y \in t^{-1}U \overset{\text{open}}{\subseteq} X$. We find some $t'x \in t^{-1}U$ since $y \in \overline{Tx}$. So $tt'x \in Tx \cap U$. $\square$

Fact A.17.76. Every flow (X, T) contains a minimal subflow.

Proof. We use Zorn's lemma: Let S be the set of all subflows of (X, T) ordered by $Y \leqslant Y' :\iff Y \supseteq Y'$. We need to show that for a chain $\langle Y_i : i \in I \rangle$, there exists a lower bound. Consider $\bigcap_{i \in I} Y_i$. This a subflow:

- It is closed as it is an intersection of closed sets.
- It is T-invariant, since each of the Y_i is.
- It is non-empty by Fact A.17.77.

$\square$

Fact A.17.77. Let X be a topological space. Then X is compact iff every family of closed sets with FIP[a] has non-empty intersection.

[a] finite intersection property, i.e. the intersection of every finite sub-family is non-empty

Proof. Note that families of closed sets correspond to families of open sets by taking complements. A family of open sets is a cover iff the corresponding family has empty intersection, and is admits a finite subcover iff the corresponding family has the FIP. $\square$

[Tutorial 12b, 2024-01-16T13:09:02]

A.10 Sheet 10

Copy from Abdelrahman

A.10.1 Exercise 2

Def skew shift flow (on $(\mathbb{R}/\mathbb{Z})^2$!)

The Bernoulli shift, $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$, is not distal. Let $x = (0)$ and $y = (\delta_{0,i})_{i \in \mathbb{Z}}$. Let $t_n \to \infty$. Then $t_n y \to (0) = t_n x$.

Proof of Fact† 4.26.38. . d and $d'(x, y) := \sup_{t \in T} d(tx, ty)$ induce the same topology. Let τ, τ' be the corresponding topologies.

$\tau \subseteq \tau'$ easy, $\tau' \subseteq \tau'$: use equicontinuity. $\square$

[Tutorial 12, 2024-01-16T12:00]

Question A.17.78. What is an example of a flow with a dense orbit that isn't minimal.

Example A.18. Consider the Bernoulli shift. $T = \mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$. (0) is a subflow. Let $\varphi\colon \mathbb{Z} \to \{0,1\}^{<\omega}$ be an enumeration of all finite binary sequences. Consider the concatenation

$$\ldots ^\frown \varphi(-2) ^\frown \varphi(-1) ^\frown \varphi(0) ^\frown \varphi(1) ^\frown \varphi(2) ^\frown \ldots$$

A.11 Sheet 11

Fact A.18.79. If A, B are topological spaces, then $f\colon A \to B$, is continuous iff $f(\overline{S}) \subseteq \overline{f(S)}$ for all $S \subseteq A$.

Proof. Suppose that f is continuous. Take $a \in \overline{S}$. Take any $f(a) \in U \overset{\text{open}}{\subseteq} B$. $f^{-1}(U)$ is open and $f^{-1}(U) \ni a$. So there exists $s \in S$ such that $s \in f^{-1}(U)$ and $f(s) \in U$.

On the other hand suppose $f(\overline{S}) \subseteq \overline{f(S)}$ for all $S \subseteq A$. It suffices to show that preimages of closed sets are closed. Let $V \overset{\text{closed}}{\subseteq} B$. Then $f(\overline{f^{-1}(V)}) \subseteq \overline{ff^{-1}(V)} \subseteq V$, hence

$$\overline{f^{-1}(V)} \subseteq f^{-1}(f(\overline{f^{-1}(V)})) \subseteq f^{-1}(V).$$

□

Fact A.18.80. Let A be compact and B Hausdorff. Let $f\colon A \to B$ be continuous and $S \subseteq A$. Then $f(\overline{S}) = \overline{f(S)}$.

Subproof. We have already shown $f(\overline{S}) \subseteq \overline{f(S)}$. Since A is compact, $f(\overline{S})$ is compact and since B is Hausdorff, compact subsets of B are closed. ■

A.11.1 Exercise 1

Let (X,T) be a flow and $G = E(X,T)$ its Ellis semigroup. Let d be a compatible metric on X.

(a) Let $f_0 \in X^X$ be a continuous function. Then $L_{f_0}\colon X^X \to X^X, f \mapsto f_0 \circ f$ is continuous.

Consider $\{f : f_0 \circ f \in U_\varepsilon(x,y)\}$. We have

$$\begin{aligned} & f_0 \circ f \in U_\varepsilon(x,y) \\ \iff & d(x, f_0(f(y))) < \varepsilon \\ \iff & f(y) \in f_0^{-1}(B_\varepsilon(x)) \\ \iff & f \in \bigcup_{\tilde{x} \in f_0^{-1}(B_\varepsilon(x))} U_{\varepsilon_{\tilde{x}}}(\tilde{x}, y) \end{aligned}$$

where $\varepsilon_{\tilde{x}}$ is such that $B_{\varepsilon_{\tilde{x}}}(\tilde{x}) \subseteq f_0^{-1}(B_\varepsilon(x))$. (This is possible since f_0 is continuous, hence $f_0^{-1}(B_\varepsilon(x))$ is open.) Clearly the RHS is open.

(b) If f_0 is not continuous, then L_{f_0} is in general not continuous: Let $X = [0,1]$, and $f_0 := \mathbb{1}_{\mathbb{Q}}$. Consider $U := U_{\frac{1}{2}}(1,1)$. Then $\{f : f_0 \circ f \in U\} = \{f : f(1) \in \mathbb{Q}\}$ is not open.

(c) Let $x_0 \in X$. The evaluation map

$$\begin{aligned} \mathrm{ev}_{x_0} \colon X^X &\longrightarrow X \\ f &\longmapsto f(x_0) \end{aligned}$$

is continuous:

Let $y \in X$ and consider $B_\varepsilon(y) \subseteq X$. By definition $\mathrm{ev}_{x_0}(B_\varepsilon(y)) = U_\varepsilon(y, x_0)$ is open.

(d) For any $x \in X$, we have $Gx = \overline{Tx}$:

By definition $G = \overline{\{x \mapsto tx : t \in T\}}$. Consider $\mathrm{ev}_x \colon X^X \to X$. X^X is compact and X is Hausdorff. Hence we can apply Fact A.18.80.

(e) Let $x_0 \neq x_1 \in X$. Then (x_0, x_1) is a proximal pair iff $d(gx_0, gx_1) = 0$ for some $g \in G$:

Let (x_0, x_1) be proximal. Consider $(\mathrm{ev}_{x_0}, \mathrm{ev}_{x_1}) \colon X^X \to X \times X$ and $d \colon X \times X \to \mathbb{R}$. Both maps are continuous. Consider $D := \{d(gx_0, gx_1) : g \in G\}$. G is compact, hence D is compact. D contains elements arbitrarily close to 0 and D is closed, so $0 \in D$.

On the other hand let $g \in G$ be such that $d(gx_0, gx_1) = 0$. We want to show that (x_0, x_1) is proximal.

As $gx_0 = gx_1$, we have that there eixsts $\varepsilon > 0$ such that $g \in U_\varepsilon(gx_0, x_1) \cap U_\varepsilon(gx_1, x_0)$ As $g \in \overline{T}$ for all $\varepsilon > 0$, there exists $t \in T$ with $t \in U_\varepsilon(gx_0, x_1) \cap U_\varepsilon(gx_1, x_0)$. Hence $d(tx_1, gx_0) < \varepsilon$ and $d(tx_0, gx_1) < \varepsilon$.

(f) (X, T) contains a minimal subflow:

We apply Zorn's lemma. It suffices to show that a chain of subflows $X \supseteq X_1 \supseteq X_2 \supseteq \ldots$ has a limit. We claim that $(\bigcap_n X_n, T)$ is a subflow, i.e. $\bigcap_n X_n$ is T-invariant. Indeed, since all the X_n are T-invariant, we have $T(\bigcap_n X_n) \subseteq \bigcap_n TX_n \subseteq \bigcap_n X_n$.

It is clear that $\bigcap_n X_n$ is closed. Since X is compact the intersection is also non-empty.

(g) Show that if T is a compact metrisable topological flow, then (X, T) is equicontinuous.

Suppose that (X, T) is not equicontinuous. Then there exists $\varepsilon > 0$ such that

$$\forall \delta > 0.\ \exists x, y \in X, t \in Y.\ d(x, y) < \delta \wedge d(tx, ty) \geqslant \varepsilon.$$

Take $\delta_n = \frac{1}{n}$. Choose bad x_n, y_n, t_n. Since X and T are compact, wlog. $x_n \to x'$, $y_n \to y'$, $t_n \to t'$. So $d(t'x', t'y') > \varepsilon$, but $x' = y'$ ↯

[Tutorial 14, 2024-01-30]

A.12 Sheet 12

A.12.1 Exercise 1

Let $\mathrm{LO}(\mathbb{N}) \overset{\text{closed}}{\subseteq} 2^{\mathbb{N}\times\mathbb{N}}$ denote the set of linear orders on $\mathbb{N}$.

Let $S \subseteq \mathrm{LO}(\mathbb{N})$ be the set of orders having a least element and such that every element has an immediate successor.

- S is Borel in $\mathrm{LO}(\mathbb{N})$:

  Let $M_n \subseteq \mathrm{LO}(\mathbb{N})$ be the set of orders with minimal element n. Let $I_{n,m} \subseteq \mathrm{LO}(\mathbb{N})$ be the set of orders such that m is the immediate successor of n.

  Clearly $S = \left(\bigcap_n \bigcup_{m\neq n} I_{n,m}\right) \cap \bigcup_n M_n$, so it suffices to show that M_n and $I_{n,m}$ are Borel. It is $M_n = \bigcap_{m\neq n}\{<: m \nprec n\}$ and $I_{n,m} = \{<: n \prec m\} \cap \bigcap_i \{<: n \preceq i \preceq m \implies n = i \vee n = m\}$.

- Give an example of an element of S which is not well-ordered:

  Consider $\{1 - \frac{1}{n} : n \in \mathbb{N}^+\} \cup \{1 + \frac{1}{n} : n \in \mathbb{N}^+\} \subseteq \mathbb{R}$ with the order $<_{\mathbb{R}}$. This is an element of S, but $\{x \in S : x \geqslant 1\}$ has no minimal element, hence it is not well-ordered.

A.12.2 Exercise 2

Recall the definition of the circle shift flow $(\mathbb{R}/\mathbb{Z}, \mathbb{Z})$ with parameter $\alpha \in \mathbb{R}$, $1 \cdot x := x + \alpha$.

- If $\alpha \notin \mathbb{Q}$, then $(\mathbb{R}/\mathbb{Z}, \mathbb{Z})$ is minimal:

  This is known as Dirichlet's Approximation Theorem.

- Consider $\mathbb{R}/\mathbb{Z}$ as a topological group. Any subgroup H of $\mathbb{R}/\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$ or of the form $H = \{x \in \mathbb{R}/\mathbb{Z} | mx = 0\}$ for some $m \in \mathbb{Z}$.

  If H contains an irrational element α, then it is dense by the previous point.

  Suppose that $H \subseteq \mathbb{Q}/\mathbb{Z}$. Let D be the set of denominators of elements of H written as irreducible fractions. If D is finite, then $H = \{x \in \mathbb{R}/\mathbb{Z} : lcm(D)x = 0\}$. Otherwise H is dense, as it contains elements of arbitrarily large denominator.

A.12.3 Exercise 3

(a) (X, T) is distal iff it does not have a proximal pair, i.e. $a \neq b$, c such that $t_n \in T$, $t_n a, t_n b \to c$.

Equivalently, for all a, b there exists an ε, such that for all $t \in T$, $d(ta, tb) > \varepsilon$.

(b) TODO

A.12.4 Exercise 4

Let X be a metrizable topological space and let $K(X) := \{K \subseteq X : K \text{ compact}\}$.

The Vietoris topology has a basis given by $\{K \subseteq U\}$, U open (type 1) and $\{K : K \cap U \neq \emptyset\}$, U open (type 2).

The Hausdorff metric on $K(X)$, $d_H(K, L)$ is the smallest ε such that $K \subseteq B_\varepsilon(L) \wedge L \subseteq B_\varepsilon(K)$. This is equal to the maximal point to set distance, $\max_{a \in A} d(a, B)$.

On previous sheets, we checked that d_H is a metric. If X is separable, then so is $K(X)$.

Fact A.18.81. Let (X, d) be a complete metric space. Then so is $(K(X), d_H)$.

Proof of Fact A.18.81. We need to show that $(K(X), d_H)$ is complete.

Let $(K_n)_{n<\omega}$ be Cauchy in $(K(X), d_H)$. Wlog. $K_n \neq \emptyset$ for all n.

Let $K = \{x \in X : \forall x \in U \overset{\text{open}}{\subseteq} X.\ U \cap K_n \neq \emptyset \text{ for infinitely many } n\}$.

Equivalently, $K = \{x : x \text{ is a cluster point of some subsequence } (x_n) \text{ with } x_n \in K_n \text{ for all } K_n\}$.

(A cluster point is a limit of some subsequence).

Claim A.18.81.1. *$K_n \to K$.*

Proof of Claim A.18.81.1. Note that K is closed (the complement is open).

Claim A.18.81.1.1. *$K \neq \emptyset$.*

Subproof. As (K_n) is Cauchy, there exists a sequence (x_n) with $x_n \in K_n$ such that there exists a subsequence (x_{n_i}) with $d(x_{n_i}, x_{n_{i+1}}) < \frac{1}{2^{i+1}}$.

Let $n_0, n_1, \ldots$ be such that $d_H(K_a, K_b) < 2^{-i-1}$ for $a, b \geqslant n_i$.

Pick $x_{n_0} \in K_{n_0}$. Then let $x_{n_{i+1}} \in K_{n_{i+1}}$ be such that $d(x_{n_i}, x_{n_{i+1}})$ is minimal.

Then $x_{n_i} \xrightarrow{i \to \infty} x$ and we have $x \in K$. ■

Claim A.18.81.1.2. *K is compact.*

Subproof. We show that K is complete and totally bounded. Since K is a closed subset of a complete space, it is complete.

So it suffices to show that K is totally bounded. Let $\varepsilon > 0$. Take N such that $d_H(K_i, K_j) < \varepsilon$ for all $i, j \geqslant N$.

Cover K_N with finitely many ε-balls with centers z_i.

Take $x \in K$. Then the ε-ball around x intersects K_j for some $j \geqslant N$, so there exists z_i such that $d(x, z_i) < 3\varepsilon$.

Note that a subset of a bigger space is totally bounded iff it is totally bounded in itself. ■

Now we show that $K_n \to K$ in $K(X)$.

Let $\varepsilon > 0$. Take N such that for all $m, n \geqslant N$, $d_H(K_m, K_n) < \frac{\varepsilon}{2}$. We'll first show that $\delta(K, K_n) < \varepsilon$ for all $n > N$.

Let $x \in K$. Take (x_{n_i}) with $x_{n_i} \in K_{n_i}, x_{n_i} \to x$. Then for large i, we have $n_i \geqslant N$ and $d(x_{n_i}, x) < \frac{\varepsilon}{2}$. Take $n \geqslant N$. Then there exists $y_n \in K_n$ with $d(y_n, x_{n_i}) < \frac{\varepsilon}{2}$. So $d(x, y_n) < \varepsilon$.

Now show that $\delta(K_n, K) < \varepsilon$ for all $n \geqslant N$.

Take $y \in K_n$. Show that $d(y, K) < \varepsilon$. To do this, construct a sequence of $y_{n_i} \in K_{n_i}$ starting with y such that $d(y_{n_i}, y_{n_{i+1}}) < \frac{\varepsilon}{2^{i+2}}$. (same trick as before). □

□

Fact A.18.82. If X is compact metrisable, then so is $K(X)$.

Proof. We have just shown that X is complete. So it suffices to show that it is totally bounded.

Let $\varepsilon > 0$. Cover X with finitely many ε-balls. Let F be the set of the centers of these balls.

Consider $\mathcal{P}(F) \setminus \{\emptyset\}$. Clearly $\{B_x^{d_H} : x \in \mathcal{P}(F) \setminus \{\emptyset\}\}$ is a finite cover of $K(X)$. □

A.13 Additional Tutorial

[Tutorial 15, 2024-01-31]

The following is not relevant for the exam, but aims to give a more general picture.

Let X be a topological space and let $\mathcal{F}$ be a filter on X.

$x \in X$ is a limit point of $\mathcal{F}$ iff the neighbourhood filter $\mathcal{N}_x$, all sets containing an open neighbourhood of x, is contained in $\mathcal{F}$.

Fact A.18.83. X is Hausdorff iff every filter has at most one limit point.

Proof. Neighbourhood filters are compatible iff the corresponding points can not be separated by open subsets. □

Fact A.18.84. X is (quasi-) compact iff every ultrafilter converges.

Proof. Suppose that X is compact. Let $\mathcal{U}$ be an ultrafilter. Consider the family $\mathcal{V} = \{\overline{A} : A \in \mathcal{U}\}$ of closed sets. By the FIP we get that there exist $c \in X$ such that $c \in \overline{A}$ for all $A \in \mathcal{U}$. Let N be an open neighbourhood of c. If $N^c \in \mathcal{U}$, then $c \in N^c$ ↯ So we get that $N \in \mathcal{U}$.

Let $\{V_i : i \in I\}$ be a family of closed sets with the FIP. Consider the filter generated by this family. We extend this to an ultrafilter. The limit of this ultrafilter is contained in all the V_i. □

Let X, Y be topological spaces, $\mathcal{B}$ a filter base on X, $\mathcal{F}$ the filter generated by $\mathcal{B}$ and $f\colon X \to Y$. Then $f(\mathcal{B})$ is a filter base on Y, since $f(\bigcap A_i) \subseteq \bigcap f(A_i)$. We say that $\lim\limits_{\mathcal{F}} f = y$, if $f(\mathcal{F}) \to y$.

Equivalently $f^{-1}(N) \in \mathcal{F}$ for all neighbourhoods N of y.

In the lecture we only considered $X = \mathbb{N}$. If $\mathcal{B}$ is the base of an ultrafilter, so is $f(\mathcal{B})$.

Fact A.18.85. Let X be a topological space and let Y be Hausdorff. Let $f, g\colon X \to Y$ be continuous. Let $A \subseteq X$ be dense such that $f|_A = g|_A$. Then $f = g$.

Proof. Consider $(f, g)^{-1}(\Delta) \supseteq A$. The LHS is a dense closed set, i.e. the entire space. □

We can uniquely extend a continuous $f\colon X \to Y$ to a continuous $\overline{f}\colon \beta X \to Y$ by setting $\overline{f}(\mathcal{U}) := \lim\limits_{\mathcal{U}} f$.

I missed the last 5 minutes

B Facts

B.1 Topological Dynamics

Fact B.0.86 ([Sua])**.** Let H be a topological group and $G \subseteq H$ a subgroup. Then $\overline{G}$ is a topological group.

Moreover if H is Hausdorff and G is abelian, then is $\overline{G}$ is abelian.

Proof. Let $g, h \in \overline{G}$. We need to show that $g \cdot h \in \overline{G}$. Take some open neighbourhood $g \cdot h \in U \overset{\text{open}}{\subseteq} H$. Let $V \overset{\text{open}}{\subseteq} H \times H$ be the preimage of U under $(a, b) \mapsto a \cdot b$. Let $A \times B \subseteq V$ for some open neighbourhoods of g resp. h. Take $g' \in A \cap G$ and $h' \in B \cap G$. Then $g'h' \in U \cap G$, hence $U \cap G \neq \emptyset$.

Similarly one shows that $\overline{G}$ is closed under inverse images.

Now suppose that H is Hausdorff and G is abelian. Consider $f\colon (g, h) \mapsto [g, h]$[45]. Clearly this is continuous. Since G is abelian, we have $f(G \times G) = \{1\}$. Since H is Hausdorff, $\{1\}$ is closed, so

$$\{1\} = \overline{f(G \times G)} \supseteq f(\overline{G \times G}) = f(\overline{G} \times \overline{G}).$$

$\square$

[45]Recall that the **commutator** is $[g, h] \coloneqq g^{-1}hgh^{-1}$.

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